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Analogues of Miyachi and Cowling-Price theorems for the generalized Dunkl transform

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ABSTRACT

In this paper,we consider the generalized Dunkl transform which satisfies some uncertainty principles similar to the Euclidean Fourier transform. A generalization of Cowling-Price's theorem, Miyachi's theorem are obtained for the generalized Dunkl transform.The techniques of the proofs are based on the properties of the generalized Dunkl kernel, the relation between the generalized Dunkl transform with the classical Dunkl transform. The results of this paper are new, and they have novelty and generalize some results exist in the literature.

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1. INTRODUCTION AND PRELIMINARIES

There are many theorems which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot simultaneously be very small at infinity. This principle has several version which were proved by M.G. Cowling and J.F. Price [2], Miyachi [3]. We refer to [4–9] for more work in this direction. In this paper we study an analogue of Cowling-Price's theorem, and Miyachi's theorem for the generalized Dunkl transform. The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the generalized Dunkl transform. In Section 3 we give an analogue of Cowling-Price's theorem and Miyachi's theorem. Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper we denote by

- $m_\alpha = \dfrac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}$ where $\alpha > \dfrac{-1}{2}$.
- $E(\mathbb{R})$ the space of $\mathcal{C}^\infty$ on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. That is the topology defined by semi-norms

$$P_{a,m}(f) = sup_{x\in[-a,a]} \sum_{k=0}^{m} \mid \frac{d^k}{dx^k} f(x) \mid, \quad a>0, \quad m=0,1,...$$

- $D_a(\mathbb{R})$, the space of $\mathcal{C}^\infty$ function on $\mathbb{R}$, which are supported in $[-a,a]$, equipped with the topology induced by $E(\mathbb{R})$
- $D(\mathbb{R}) = \bigcup_{a>0} D_a(\mathbb{R})$, endowed with inductive limit topology.
- $E_n(\mathbb{R})$(resp.$D_n(\mathbb{R})$) stand for the subspace of $E(\mathbb{R})$(resp.$D(\mathbb{R})$) consisting of functions f such that

$$f(0) = ... = f^{(2n-1)}(0) = 0.$$

For $a>0$, put

$$D_{a,n}(\mathbb{R}) = D_a(\mathbb{R}) \cap E_n(\mathbb{R}).$$

In this section we recapitulate some facts about harmonic analysis related to the generalized Dunkl transform $\mathcal{F}_{\alpha,n}$. We cite here, as briefly as possible, some properties. For more details we refer to [1, 10].

$$\Delta_{\alpha,n} f(x) = f'(x) + (\alpha+\frac{1}{2})\frac{f(x)-f(-x)}{x} - 2n\frac{f(-x)}{x},$$

The one-dimensional generalized Dunkl kernel $\Lambda_{\alpha,n}$ is defined by

$$\Lambda_{\alpha,n}(x,\lambda) = a_{\alpha+2n} x^{2n} \int_{-1}^{1} (1-t^2)^{\alpha+2n-\frac{1}{2}}(1+t)e^{i\lambda xt}dt,$$

where $a_{\alpha+2n} = \dfrac{2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})}$.

$\Lambda_{\alpha,n}(.,\lambda)$ satisfies the differential-difference equation

$$\Delta_{\alpha,n}\Lambda_{\alpha,n}(.,\lambda) = i\lambda\Lambda_{\alpha,n}(.,\lambda).$$

For all $m = 0, 1, ...$

$$\mid \frac{\partial^m}{\partial\lambda^m}\Lambda_{\alpha,n}(x,\lambda) \mid \leq |x|^{2n+m} e^{|Im\ \lambda||x|}. \tag{1}$$

The generalized Dunkl transform of a function $f \in D_n(\mathbb{R})$ is defined by

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Lambda_{\alpha,n}(x,-\lambda)d\mu_\alpha(x), \quad \lambda \in \mathbb{C},$$

where

$$d\mu_\alpha(x) = |x|^{2\alpha+1}dx. \tag{2}$$

For all $f \in D_n(\mathbb{R})$, we have the inversion formula

$$f(x) = m_{\alpha+2n}\int_{\mathbb{R}} \mathcal{F}_{\alpha,n}(f)(\lambda)\Lambda_{\alpha,n}(x,\lambda)d\mu_{\alpha+2n}(\lambda). \tag{3}$$

For every $f \in D_n(\mathbb{R})$, we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 d\mu_\alpha(x) = m_{\alpha+2n}\int_{\mathbb{R}} |\mathcal{F}_{\alpha,n}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda). \tag{4}$$

We define the function $N(.,s)$, $s>0$ as follows

$$N(x,s) = m_{\alpha+2n}\int_{\mathbb{R}} e^{-ry^2}\Lambda_{\alpha,n}(x,y)d\mu_{\alpha+2n}(y)\ x \in \mathbb{R}.$$

We denote by $L^p_\alpha(\mathbb{R})$, $1 \leq p \leq +\infty$ the space of measurable functions on $\mathbb{R}$ such that

$$\|f\|_{L^p_\alpha(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}} < +\infty, \ \ if\ 1 \leq p < +\infty,$$

$$\|f\|_{L^\infty_\alpha(\mathbb{R})} = ess \sup_{x\in\mathbb{R}} |f(x)| < +\infty, \ \ if\ p = \infty.$$

$L^p_{\alpha,n}(\mathbb{R})$, $1 \leq p \leq \infty$, be the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{L^p_{\alpha,n}(\mathbb{R})} = \|x^{-2n}f\|_{L^p_{\alpha+2n}(\mathbb{R})} < \infty.$$

Proposition 1..1 *For all $f \in L^1_{\alpha,n}(\mathbb{R})$ we have*

$$\|\mathcal{F}_{\alpha,n}(f)\|_{L^\infty_{\alpha+2n}(\mathbb{R})} \leq \|f\|_{L^1_{\alpha,n}(\mathbb{R})}. \tag{5}$$

Proof.

$$\begin{aligned} |\mathcal{F}_{\alpha,n}(f)(\lambda)| &= |\int_{\mathbb{R}} f(x)\Lambda_{\alpha,n}(x,-\lambda)d\mu_\alpha(x)| \\ &\leq \int_{\mathbb{R}} |f(x)||\Lambda_{\alpha,n}(x,-\lambda)|d\mu_\alpha(x), \end{aligned}$$

it follows from (1) that

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|.|\Lambda_{\alpha,n}(x,-\lambda)|d\mu_\alpha(x) &\leq \int_{\mathbb{R}} |f(x)|x^{2n}d\mu_\alpha(x) \\ &\leq \int_{\mathbb{R}} \frac{|f(x)|}{x^{2n}}d\mu_{\alpha+2n}(x) \\ &= \|f\|_{L^1_{\alpha,n}}(\mathbb{R}), \end{aligned}$$

which proves the desired result. ■

The generalized Dunkl intertwining operator on $\mathbb{R}$ is defined by

$$V_{\alpha,n}(f)(x) = a_{\alpha+2n}x^{2n}\int_{-1}^{1} f(tx)(1-t^2)^{\alpha-\frac{1}{2}}(1+t)dt. \tag{6}$$

The dual of the generalized Dunkl intertwining on $D_n(\mathbb{R})$ is defined by

$${}^tV_{\alpha,n}(f)(y) = a_{\alpha+2n}\int_{|x|\geq|y|} f(x)sgn(x)(x^2-y^2)^{\alpha+2n-\frac{1}{2}}(x+y)\frac{dx}{x^{2n}}, \quad y \in \mathbb{R}. \tag{7}$$

In the next we denote by

$$dv^\alpha_y(x) = a_\alpha sgn(x)(x^2-y^2)^{\alpha-\frac{1}{2}}(x+y)1_{[|y|,+\infty[}(|x|)dx. \tag{8}$$

The generalized Dunkl intertwining operator $V_{\alpha,n}$ and its dual ${}^tV_{\alpha,n}$ are related with the following formula

$$\int_{\mathbb{R}} V_{\alpha,n}(f)(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} f(y)^tV_{\alpha,n}g(y)dy, \tag{9}$$

where $f \in E(\mathbb{R})$ and $g \in D(\mathbb{R})$.

Proposition 1..2 *${}^tV_{\alpha,n}$ is a bounded operator from $L^1_{\alpha,n}(\mathbb{R})$ to $L^1(\mathbb{R})$ where $L^1(\mathbb{R})$ is the space of Lebesgue-integrable functions.*

2. COWLING-PRICE'S THEOREM FOR THE GENERALIZED DUNKL TRANSFORM

Theorem 2..1 *Let f be a measurable function on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \frac{e^{apx^2}|f(x)|^p}{x^{2np}(1+|x|)^k}|x|^{2\alpha+1}dx < \infty \tag{10}$$

and

$$\int_{\mathbb{R}} \frac{e^{bq\xi^2}|\mathcal{F}_{\alpha,n}(f)(\xi)|^q}{(1+|\xi|)^m}d\xi < \infty, \tag{11}$$

for some constants $a, b > 0,\ k > 0, m > 1$ and $1 \leq p, q \leq +\infty$.

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$, then $f(x) = Q(x)N(x,b)$ where Q is a polynomial with $degQ \leq \frac{m-1}{q}$. Especially, if

$$k \leq 2\alpha + 4n + 2 + p\min\{\frac{k}{p} + \frac{2\alpha+4n+1}{p'}, \frac{m-1}{q}\},$$

then $f = 0$ almost everywhere. Furthermore, if $m \in]1, 1+q]$ and $k > 2\alpha + 4n + 2$, then f is a constant multiple of $N(.,b)$.

iii) If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4}a[$ all functions of the form $f(x) = P(x)e^{bx^2}$ satisfy (10) and (11).

Proof. It follows from (10) that $f \in L^1_{\alpha,n}$ and $\mathcal{F}_{\alpha,n}(f)(\xi)$ exists for all $\xi \in \mathbb{R}$. Moreover, it has an entire holomorphic extension on $\mathbb{C}$ satisfying for some $s > 0$,

$$|\mathcal{F}_{\alpha,n}(f)(z)| \leq Ce^{\frac{Imz^2}{4a}}(1 + |Imz|)^s.$$

By (1) we have for all $z = \xi + i\eta \in \mathbb{C}$,

$$|\mathcal{F}_{\alpha,n}(f)(z)| \leq \int_{\mathbb{R}} |f(x)||\Lambda_{\xi+i\eta,\alpha,n}(x)||x|^{2\alpha+1}dx \quad (12)$$

$$\leq e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} \frac{e^{ax^2}|f(x)|}{x^{2n}(1+|x|)^{\frac{k}{p}}} x^{4n}(1+|x|)^{\frac{k}{p}} e^{-a(x-\frac{\eta}{2a})^2}|x|^{2\alpha+1}dx. \quad (13)$$

A combination of (10) and Hlder inequality shows that

$$\begin{aligned}
|\mathcal{F}_{\alpha,n}(f)(\xi+i\eta)| &\leq Ce^{\frac{\eta^2}{4a}} \left(\int_{\mathbb{R}} (1+|x|)^{\frac{kp'}{p}} e^{-ap'(x-\frac{\eta}{2a})^2}|x|^{2\alpha+4n+1}dx\right)^{\frac{1}{p'}} \\
&\leq Ce^{\frac{\eta^2}{4a}} \left(\int_{\mathbb{R}} (1+|x|)^{\frac{kp'}{p}+2\alpha+4n+1} e^{-ap'(x-\frac{\eta}{2a})^2}dx\right)^{\frac{1}{p'}} \\
&\leq Ce^{\frac{\eta^2}{4a}} \left(\int_0^{\infty} (1+|x|)^{\frac{kp'}{p}+2\alpha+4n+1} e^{-ap'(x-\frac{\eta}{2a})^2}dx\right)^{\frac{1}{p'}} \\
&\leq Ce^{\frac{\eta^2}{4a}}(1+|\eta|)^{\frac{k}{p}+\frac{2\alpha+4n+1}{p'}}.
\end{aligned}$$

If $ab = \frac{1}{4}$, then

$$|\mathcal{F}_{\alpha,n}(f)(\xi+i\eta)| \leq Ce^{b\eta^2}(1+|\eta|)^{\frac{k}{p}+\frac{2\alpha+4n+1}{p'}}.$$

We put $g(z) = e^{bz^2}\mathcal{F}_{\alpha,n}(f)(z)$, then

$$|g(z)| \leq Ce^{b|Rez|^2}(1+|Imz|)^{\frac{k}{p}+\frac{2\alpha+4n+1}{p'}}.$$

It follows from (11) that

$$\int_{\mathbb{R}} \frac{|g(z)|^q}{(1+|\xi|)^m} d\xi < \infty.$$

Lemma 2..2 *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a|Rez|^2}(1+|Imz|)^l$$

for some $l > 0$, $a > 0$ and

$$\int_{\mathbb{R}} \frac{|h(x)|^q}{(1+|x|)^m}|Q(x)|dx < \infty$$

for some $q \geq 1$, $m > 1$ and $Q \in P(\mathbb{R})$. Then h is a polynomial with $degh \leq min\{l, \frac{m-M-1}{q}\}$ and, if $m \leq q + M + 1$, then h is a constant.

From this Lemma g is a polynomial, we say P_b with $degP_b \leq min\{\frac{kp'}{p}+\frac{2\alpha+4n+1}{p'}, \frac{m-1}{q}\}$. Then $\mathcal{F}_{\alpha,n}(f)(x) = P_b(x)e^{-bx^2}$ then,

$$f(x) = Q_b(x)N(x,b)$$

where $degP_b = degQ_b$. Therefore , nonzero f satisfies (10) provided that

$$k > 2\alpha + 4n + 2 + p\min\left\{\frac{kp'}{p} + \frac{2\alpha+4n+1}{p'}, \frac{m-1}{q}\right\}.$$

If $m < q + 1$, by Lemma 1 we have g is a constant and $\mathcal{F}_{\alpha,n}(f)(x) = Ce^{-bx^2}$ and $f(x) = Cx^{2n}N(x,b)$. If $m > 1$ and $k > 2\alpha + 4n + 2$, these functions satisfy (10) and (11), which proves (ii).

If $ab > \frac{1}{4}$, then we can find positive constants a_1 and b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\mathcal{F}_{\alpha,n}(f)$ also satisfy (11) with a and b replaced by a_1 and b_1 respectively. Then $\mathcal{F}_{\alpha,n}(f)(x) = P_{b_1}(x)e^{-b_1x^2}$. $\mathcal{F}_{\alpha,n}(f)$ cannot satisfy (11) unless $P_{b_1} = 0$, which implies that $f = 0$, this proves (i). If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4a}[$, the functions of the form $f(x) = x^{2n}P(x)N_k(x,\delta)$, where P is a polynomial on $\mathbb{R}$, satisfy (10) and (11). This proves (iii).

■

3. MIYACHI'S THEOREM FOR THE GENERALIZED DUNKL TRANSFORM

Theorem 3..1 *Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2}f \in L^p_{\alpha,n}(\mathbb{R}) + L^q_{\alpha,n}(\mathbb{R}) \tag{14}$$

and

$$\int_{\mathbb{R}} \log^+ \frac{|\mathcal{F}_{\alpha,n}(f)(\xi)e^{b\xi}|}{\lambda} d\xi < \infty, \tag{15}$$

for some constants $a, b, \lambda > 0$ and $1 \leq p, q \leq +\infty$.

(i) if $ab > \frac{1}{4}$ then $f = 0$ almost everywhere.

(ii) if $ab = \frac{1}{4}$ then $f = cN(.,b)$ with $|c| \leq \lambda$.

(iii) if $ab > \frac{1}{4}$ then for all $\delta \in]b, \frac{1}{4}[$, all functions of the form $f(x) = P(x)N(x,\delta)$, where P is a polynomial on $\mathbb{R}$ satisfy (14) and (15).

To prove this result, we need the following lemmas.

Lemma 3..2 *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ae^{B|Rez|^2},$$

and

$$\int_{\mathbb{R}} \log^+ |h(y)|dy < \infty, \tag{16}$$

for some constants A and B. Then h is a constant.

Lemma 3..3 *Let $r \in [1, +\infty], a > 0$. Then for $g \in L^r_{\alpha,n}(\mathbb{R})$ there exist $c > 0$ such that*

$$\| e^{ax^2}\ {}^tV_{\alpha,n}(e^{-ay^2}g) \|_r \leq c \| g \|_{r,\alpha,n} .$$

Proof. From the hypothesis, it follows that e^{-ay^2} belongs to $L^1_{\alpha,n}(\mathbb{R})$. Then by proposition 2, ${}^tV_{\alpha,n}(e^{-ay^2}g)$ is defined almost everywhere on $\mathbb{R}$. Here we consider two cases:

i) If $r \in [1, +\infty[$ then

$$\begin{aligned}\| e^{ax^2}\, {}^tV_{\alpha,n}(e^{-ay^2}g) \|_r^r &\leq \int_{\mathbb{R}} e^{arx^2}(\int_{\mathbb{R}} y^{-2n}e^{-ay^2}|g(y)|d\nu_x(y))^r dx,\\ &\leq \int_{\mathbb{R}} e^{arx^2}(\int_{\mathbb{R}} |y^{-2n}g(y)|^r d\nu_x(y))(\int_{\mathbb{R}} e^{-ar'y^2}d\nu_x(y))^{\frac{r}{r'}}dx\end{aligned}$$

where r' is the conjugate exponent for r. Since

$$\int_{\mathbb{R}} e^{-ry^2}d\nu_x(y) = Ce^{-rx^2}, \tag{17}$$

for $r > 0$ it follow from (17) that

$$\begin{aligned}\| e^{ax^2}\, {}^tV_{\alpha,n}(e^{-ay^2}g) \|_r^r &\leq C\int_{\mathbb{R}} {}^tV_{\alpha,n}(|g|^r)(x)dx,\\ &= C\int_{\mathbb{R}} |g(x)|^r|x|^{2\alpha+1}dx < \infty.\end{aligned}$$

ii) If $r = \infty$ then it follow from (17) that

$$\begin{aligned}\| e^{ax^2}\, {}^tV_{\alpha,n}(e^{-ay^2}g) \|_r &\leq e^{ax^2}\, {}^tV_{\alpha,n}(e^{-ay^2})(x)\|g\|_{\alpha,n,\infty}\\ &= C\|g\|_{\alpha,n,\infty}.\end{aligned}$$

■

Lemma 3..4 *Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2}f \in L^p_{\alpha,n}(\mathbb{R}) + L^q_{\alpha,n}(\mathbb{R}) \tag{18}$$

for some $a > 0$. Then for all $z \in \mathbb{C}$, the integral

$$\mathcal{F}_{\alpha,n}(f)(z) = \int_{\mathbb{R}} f(x)\Lambda_{z,\alpha,n}(-x)|x|^{2\alpha+1}dx$$

is well defined. $\mathcal{F}_{\alpha,n}(f)(z)$ is entire and there exist $C > 0$ such that for all $\xi,\ \eta \in \mathbb{R}$,

$$|\mathcal{F}_{\alpha,n}(f)(\xi + i\eta)| \leq Ce^{\frac{\eta^2}{4a}}. \tag{19}$$

Proof. From (1) and Hlder's inequality we have the first assertion. For (19) using (18) we have $f \in L^1_{\alpha,n}(\mathbb{R})$ and ${}^tV_{\alpha,n}(f) \in L^1_{\alpha,n}(\mathbb{R})$. for all $\xi,\ \eta \in \mathbb{R}$,

$$\mathcal{F}_{\alpha,n}(f)(\xi + i\eta) = \int_{\mathbb{R}} {}^tV_{\alpha,n}(f)(x)e^{-ix(\xi+i\eta)}dx$$

$$\begin{aligned}|\mathcal{F}_{\alpha,n}(f)(\xi + i\eta)| &\leq e^{\frac{\eta^2}{4a}}\int_{\mathbb{R}} e^{ax^2}\left|{}^tV_{\alpha,n}(f)(x)\right|e^{-ax^2+x\eta-\frac{\eta^2}{4a}}dx\\ &\leq e^{\frac{\eta^2}{4a}}\int_{\mathbb{R}} e^{ax^2}\left|{}^tV_{\alpha,n}(f)(x)\right|e^{-a\left(x-\frac{\eta}{2a}\right)^2}dx.\end{aligned}$$

From (18) we can deduce that there exist $u \in L^p_{\alpha,n}(\mathbb{R})$ and $v \in L^q_{\alpha,n}(\mathbb{R})$ such that

$$f(x) = e^{-ax^2}u(x) + e^{-ax^2}v(x),$$

by Lemma 3 we have

$$\int_{\mathbb{R}} e^{ax^2}\left|{}^tV_{\alpha,n}(f)(x)\right|e^{-a\left(x-\frac{\eta}{2a}\right)^2}dx \leq C\left(\|u\|_{p,\alpha,n} + \|v\|_{q,\alpha,n}\right) < \infty,$$

which proves the Lemma. ■

Proof of Theorem

- If $ab > \frac{1}{4}$. Let h be a function on $\mathbb{C}$ defined by

$$h(z) = e^{\frac{z^2}{4a}} \mathcal{F}_{\alpha,n}(f)(z).$$

h is entire function on $\mathbb{C}$, it follows from (19) that

$$\forall \xi \in \mathbb{R},\ \forall \eta \in \mathbb{R}\ |h(\xi + i\eta)| \le Ce^{\frac{\xi^2}{4a}}. \tag{20}$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{R}} \log^+ |h(y)|\, dy &= \int_{\mathbb{R}} \log^+ \left| e^{\frac{y^2}{4a}} \mathcal{F}_{\alpha,n}(f)(y) \right| dy \\ &= \int_{\mathbb{R}} \log^+ \frac{\left| e^{by^2} \mathcal{F}_{\alpha,n}(f)(y) \right|}{\lambda} \lambda e^{\left(\frac{1}{4a} - b\right) y^2} dy \\ &\le \int_{\mathbb{R}} \log^+ \frac{\left| e^{by^2} \mathcal{F}_{\alpha,n}(f)(y) \right|}{\lambda} dy + \int_{\mathbb{R}} \lambda e^{\left(\frac{1}{4a} - b\right) y^2} dy \end{aligned}$$

because $\log_+(cd) \le \log_+(c) + d$ for all c, $d > 0$. Since $ab > \frac{1}{4}$, (15) implies that

$$\int_{\mathbb{R}} \log^+ |h(y)| dy < \infty. \tag{21}$$

An combination of (20), (21) and Lemma shows that h is a constant and

$$\mathcal{F}_{\alpha,n}(f)(y) = Ce^{-\frac{1}{4a} y^2}.$$

Since $ab > \frac{1}{4}$, (11) holds whenever $C = 0$ and the injectivity of $\mathcal{F}_{\alpha,n}$ implies that $f = 0$ almost everywhere.

- If $ab = \frac{1}{4}$. We deduce from previous case that $\mathcal{F}_{\alpha,n}(f) = Ce^{-\frac{\xi^2}{4a}}$. Then (11) holds whenever $|C| \le \lambda$. Hence $f = CN(., b)$ with $|C| \le \lambda$.

- If $ab < \frac{1}{4}$. If f is a given form, then $\mathcal{F}_{\alpha,n}(f)(y) = Q(y)e^{-\delta y^2}$ for some Q.

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