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Numerical solution of a piezoelectric contact problem

Youssef Ouafik
National School of Applied Sciences of Safi, Cadi Ayyad University, Safi, Morocco

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ABSTRACT

We consider a mathematical model, which describes a contact problem between a piezoelectric body and a conductive foundation. The linear electro-elastic constitutive law is employed to model the piezoelectric material. The process is static, the contact is frictionless and described with the normal compliance condition and an electric contact condition. Our aim is to present a detailed description of the numerical modelling of the problem. To this end, we introduce a discrete scheme, based on the finite element method to approximate the spatial variable. Then we treat the contact conditions by using a penalized approach and a version of Newton's method. Finally, we provide numerical simulations in the study of a two-dimensional example and compare the regularized problem with the original one.

Corresponding Author:

Youssef Ouafik,
National School of Applied Sciences of Safi,
Cadi Ayyad University,
Safi, Morocco.
Email: y.ouafik@uca.ma

1. INTRODUCTION

The effective conversion of the electrical energy into mechanical energy and vice versa has led the piezoelectric materials to important applications in many engineering structures such as sensors, actuators, intelligent structures, etc. Currently, there exists a considerable interest in contact problems involving piezoelectric materials, under the assumption that the foundation is electrically conductive (see, [1-6]).

The results in [1-6] concern the variational formulation of the problems and their numerical simulations. For all these references, the contact is described with a regularized electrical conductivity condition. A similar model was studied in [7] for almost perfect electric contact. The result in [7] concern the existence of solution for problem, however, no numerical analyse was studied. Here we continue this line of research and study a contact between electro-elastic body and a deformable conductive foundation. The contact is frictionless and is modelled with normal compliance condition. The analyse of the mechanical model is based on the assumption that the electric contact conditions are supposed almost perfect, our interest in this paper is to provide the numerical modelling of the contact problem supported by numerical simulations. To this end, we present a discretization of the problem and we describe details of the numerical algorithm we use.

The main novelty of our work arises from the fact that we study two problems: an original problem constructed with almost perfect electrical contact, and a regularization one constructed by considering a regularized condition on the electric field on the potential contact zone. In this paper, we provide a reliable comparison between numerical solutions of the approximate contact problem and the original one. Finally, we present numerical simulations, which validate our approximation method and give information on the mechanical behaviour of the solution.

The rest of paper is structured as follows. In Section 2 the piezoelectric contact problem is stated together with two variational formulations. A discrete scheme, based on the finite element, and the numerical algorithm used for solving the discrete problem are described in Section 3. In Section 4, we present numerical simulations in the study of a two-dimensional test problem. Finally, in Section 5 we present some conclusions and perspectives.

2. VARIATIONAL FORMULATION AND ITS REGULARISED

The physical setting is the following. A piezoelectric body occupies a regular domain $\Omega \subset \mathbb{R}^d, d = 2, 3$ with a smooth boundary $\partial\Omega = \Gamma$. We use boldface letters for vectors and tensors, such as the outward unit normal on Γ, denoted by $\boldsymbol{\nu} = (\nu_i)$. The body is submitted to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_D, Γ_N, Γ_C, on one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\Gamma_D > 0$ and $meas\Gamma_a > 0$. The body is clamped on Γ_D; therefore, the displacement field vanishes there. Moreover, we assume that a density of traction forces, denoted by $\boldsymbol{f}_N$, acts on the boundary part Γ_N. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_b is prescribed on Γ_b. In the reference configuration, the body is in contact over Γ_C with an electrically conductive foundation. We assume that its potential is maintained at φ_f. The contact is frictionless and described with the normal compliance condition. Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, i.e. $f_{,i} = \frac{\partial f}{\partial x_i}$.

We denote by $\boldsymbol{u} = (u_i) \in \mathbb{R}^d$ the displacement vector, by $\boldsymbol{\sigma} = (\sigma_{ij}) \in S^d$ the stress tensor, by $\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})) \in S^d$ the linearized strain tensor, i.e. $\varepsilon_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$, by $\boldsymbol{E}(\varphi) = -\nabla\varphi = -(\varphi_{,i}) \in \mathbb{R}^d$ the electric vector field, where $\varphi \in \mathbb{R}$ is the electric potential, and $\boldsymbol{D} = (D_i) \in \mathbb{R}^d$ by the electric displacement filed. The notation S^d stands for the space of second order symmetric tensors on $\mathbb{R}^d$. The functions $\boldsymbol{u} \in \mathbb{R}^d$ and $\varphi \in \mathbb{R}$ are the unknowns of the problem, and, for simplicity, we do not indicate the dependence the functions on the variable $\boldsymbol{x} \in \Omega \cup \Gamma$.

The body is assumed to be electro-elastic and, therefore, we use the constitutive law

$$\boldsymbol{\sigma} = \mathcal{F}\varepsilon(\boldsymbol{u}) - \mathcal{E}^T\boldsymbol{E}(\varphi) \quad in\ \Omega, \tag{1}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \beta\boldsymbol{E}(\varphi) \quad in\ \Omega. \tag{2}$$

Here $\mathcal{F} = (f_{ijkl})$, $\mathcal{E} = (e_{ijk})$ and $\beta = (\beta_{ij})$ are respectively, the elasticity, piezoelectric and permittivity tensors. $\mathcal{E}^T$ is the transpose of $\mathcal{E}$. Also the tensors $\mathcal{E}$ and $\mathcal{E}^T$ satisfy the equality $\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^T\boldsymbol{v} \quad \forall\, \boldsymbol{\sigma} \in S^d$, $\boldsymbol{v} \in \mathbb{R}^d$, and the components of the tensor $\mathcal{E}^T$ are given by $e^T_{ijk} = e_{kij}$.

Since the process is assumed static, then the equation of stress equilibrium and the equation of the quasi-stationary electric field, respectively, given by

$$Div\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad in\ \Omega, \tag{3}$$

$$div\, \boldsymbol{D} = q_0 \quad in\ \Omega. \tag{4}$$

Here " Div " and " div " denote the divergence operators for tensor and vector valued functions, i.e. $Div\, \boldsymbol{\sigma} = (\sigma_{ij,i})$, $div\, \boldsymbol{D} = (D_{i,i})$.

We turn to describe the boundary conditions, so on the $\Gamma_D \cup \Gamma_N$ portion of the boundary, we impose the following conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad on\ \Gamma_D, \tag{5}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad on\ \Gamma_N. \tag{6}$$

The boundary conditions for the electric potential can be defined in the following forms:

$$\varphi = 0 \quad on\ \Gamma_a, \tag{7}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = q_b \quad on\ \Gamma_b. \tag{8}$$

We now describe the electro-mechanical boundary conditions on the potential contact surface Γ_C. We assume that the normal displacement $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, the normal stress $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$ and the tangential stress $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$ satisfy the condition of normal compliance without friction:

$$\sigma_\nu = -c_\nu(u_\nu - g)_+\,,\ \ \boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad on\ \Gamma_C, \tag{9}$$

where c_ν is a positive constant, g is the gap between the body and the foundation and $(\,.\,)_+$ stands for the positive part so that $(u_\nu - g)_+$ represents the penetration of the body into the foundation. Moreover, there may be electrical charges on the contact surface:

$$\boldsymbol{D}.\boldsymbol{\nu} = k_e\chi_{[0,\infty)}(u_\nu - g)(\varphi - \varphi_f) \quad on\ \Gamma_C, \tag{10}$$

where $k_e > 0$ is the electrical conductivity coefficient and $\chi_{[0,\infty)}(.)$ is the characteristic function of the interval $[0,\infty)$, that is

$$\chi_{[0,\infty)}(r) = \begin{cases} 0, & r < 0, \\ 1, & r \geq 0. \end{cases} \tag{11}$$

Condition (10) describes perfect electrical contact, see [1].

We collect the above equations and conditions to obtain the following mathematical problem.

Problem P. Find a displacement field $\boldsymbol{u} \in \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} \in S^d$, an electric potential $\varphi \in \mathbb{R}$ and an electric displacement field $\boldsymbol{D} \in \mathbb{R}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{F}\varepsilon(\boldsymbol{u}) - \mathcal{E}^T\boldsymbol{E}(\varphi) \quad in\ \Omega, \tag{12}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \beta\boldsymbol{E}(\varphi) \quad in\ \Omega, \tag{13}$$

$$Div\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad in\ \Omega, \tag{14}$$

$$div\,\boldsymbol{D} = q_0 \quad in\ \Omega, \tag{15}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad on\ \Gamma_D, \tag{16}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad on\ \Gamma_N, \tag{17}$$

$$\varphi = 0 \quad on\ \Gamma_a, \tag{18}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = q_b \quad on\ \Gamma_b, \tag{19}$$

$$\sigma_\nu = -c_\nu(u_\nu - g)_+\,,\ \ \boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad on\ \Gamma_C, \tag{20}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = k_e\chi_{[0,\infty)}(u_\nu - g)(\varphi - \varphi_f) \quad on\ \Gamma_C. \tag{21}$$

We now turn to the variational formulation of Problem P, which is the starting point for the numerical modelling based on the finite element discretization. We denote in the sequel by " **.** " and ‖ **.** ‖ the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and S^d. We start by introducing the spaces $H = L^2(\Omega,\mathbb{R}^d)$, $\mathcal{K} = L^2(\Omega, S^d)$. The spaces H and $\mathcal{K}$ are Hilbert spaces equipped with the inner products $(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx$ and $(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{K}} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx$, respectively. The associated norms in H and $\mathcal{K}$ are denoted by $||\cdot||_H$ and $||\cdot||_{\mathcal{K}}$, respectively.

For the displacement and the electric potential fields, we introduce the spaces $V = \{\boldsymbol{v} \in H^1(\Omega,\mathbb{R}^d);\ \boldsymbol{v} = \boldsymbol{0}\ on\ \Gamma_D\}$ and $W = \{\psi \in H^1(\Omega);\ \psi = 0\ on\ \Gamma_a\}$. On V and W we consider the inner products and the corresponding norms given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{K}}\,,\quad ||\boldsymbol{v}||_V = ||\varepsilon(\boldsymbol{v})||_{\mathcal{K}} \ \text{ for all } \boldsymbol{u},\boldsymbol{v} \in V, \tag{22}$$

$$(\varphi,\psi)_W = (\nabla\varphi,\nabla\psi)_H\,,\quad ||\psi||_W = ||\nabla\psi||_H \ \text{ for all } \varphi,\psi \in W. \tag{23}$$

Since $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$ are positive, it follows from the Korn and the Friedrichs-Poincaré inequalities, respectively, that $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ are Hilbert spaces.

We consider the two mapping $\boldsymbol{f} \in V$ and $q \in W$, defined by

$$(\boldsymbol{f}, \boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{w}\, dx + \int_{\Gamma_N} \boldsymbol{f}_N \cdot \boldsymbol{w}\, da, \tag{24}$$

$$(q, \psi)_W = \int_\Omega q_0 \psi\, dx - \int_{\Gamma_b} q_b \psi\, da, \tag{25}$$

for all $\boldsymbol{w} \in V$ and $\psi \in W$.

Then, the variational formulation of the contact problem P obtained by multiplying the equations with the relevant test functions and performing integration by part is as follows.

Problem P_V. Find a displacement field $\boldsymbol{u} \in V$ and an electric potential $\varphi \in$ W such that

$$\big(\mathcal{F}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{w})\big)_{\mathcal{K}} + \big(\mathcal{E}^T \nabla\varphi, \varepsilon(\boldsymbol{w})\big)_{\mathcal{K}} + \int_{\Gamma_C} c_\nu (u_\nu - g)_+ w_\nu\, da = (\boldsymbol{f}, \boldsymbol{w})_V \quad \forall\, \boldsymbol{w} \in V, \tag{26}$$

$$(\beta\nabla\varphi, \nabla\psi)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}), \nabla\psi)_H + \int_{\Gamma_C} k_e \chi_{[0,\infty)}(u_\nu - g)\big(\varphi - \varphi_f\big)\psi\, da = (q, \psi)_W \quad \forall \psi \in W. \tag{27}$$

We consider the truncation of the function $\chi_{[0,\infty)}$ noted ψ_δ and it's defined by

$$\psi_\delta(r) = \begin{cases} 0 & if \quad r < 0, \\ \dfrac{r}{\delta} & if \quad 0 \le r \le \delta, \\ 1 & if \quad r > \delta. \end{cases} \tag{28}$$

δ is a small parameter which will tend to zero in the sequel.

Replacing $\chi_{[0,\infty)}$ by the function ψ_δ leads us to replacing $\int_{\Gamma_C} k_e \chi_{[0,\infty)}(u_\nu - g)\big(\varphi - \varphi_f\big)\psi\, da$ in P_V by $\int_{\Gamma_C} k_e \psi_\delta\, (u_\nu - g)\big(\varphi - \varphi_f\big)\psi\, da$ and we introduce now a regularized problem $P_{V\delta}$.

Problem $P_{V\delta}$. Find a displacement field $\boldsymbol{u}_\delta \in V$ and an electric potential $\varphi_\delta \in$ W such that

$$\big(\mathcal{F}\varepsilon(\boldsymbol{u}_\delta), \varepsilon(\boldsymbol{w})\big)_{\mathcal{K}} + \big(\mathcal{E}^T \nabla\varphi_\delta, \varepsilon(\boldsymbol{w})\big)_{\mathcal{K}} + \int_{\Gamma_C} c_\nu \big(u_{\delta_\nu} - g\big)_+ w_\nu\, da = (\boldsymbol{f}, \boldsymbol{w})_V \quad \forall\, \boldsymbol{w} \in V, \tag{29}$$

$$(\beta\nabla\varphi_\delta, \nabla\psi)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}_\delta), \nabla\psi)_H + \int_{\Gamma_C} k_e \psi_\delta\big(u_{\delta_\nu} - g\big)\big(\varphi_\delta - \varphi_f\big)\psi\, da = (q, \psi)_W \quad \forall \psi \in W. \tag{30}$$

3. NUMERICAL APPROACH

We now present a discrete approximation of Problems P_V and $P_{V\delta}$. First, we consider two finite dimensional spaces $V^h \subset V$ and $W^h \subset W$ approximating the spaces V and W, respectively, in which $h > 0$ denotes the spatial discretization parameter. In the numerical simulations presented in the next section, V^h and W^h consist of continuous and piecewise affine functions, that is,

$$V^h = \Big\{\boldsymbol{w}^h \in \big[C(\overline{\Omega})\big]^d\,;\ \boldsymbol{w}^h{}_{/Tr} \in\ [P_1(Tr)]^d\ \ Tr \in\ T^h, \qquad \boldsymbol{w}^h = \boldsymbol{0}\ \ on\ \ \Gamma_D\Big\}, \tag{31}$$

$$W^h = \Big\{\psi^h \in \big[C(\overline{\Omega})\big]\,;\ \psi^h{}_{/Tr} \in [P_1(Tr)]\ \ Tr \in\ T^h, \qquad \psi^h = 0\ \ on\ \ \Gamma_a\Big\}, \tag{32}$$

where Ω is assumed to be a polygonal domain, T^h denotes a finite element triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to one in Tr.

The discrete approximation of Problem P_V is the following.

Problem P_V^h. Find a discrete displacement $\boldsymbol{u}^h \in V^{\boldsymbol{h}}$ and a discrete electric potential $\varphi^h \in \mathrm{W}^{\boldsymbol{h}}$ such that

$$\left(\mathcal{F}\varepsilon(\boldsymbol{u}^h),\varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}}+\left(\mathcal{E}^T\nabla\varphi^h,\varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}}+\int_{\Gamma_C}c_\nu(u_\nu^h-g^h)_+w_\nu^h\,da = (\boldsymbol{f}^h,\boldsymbol{w}^h)_V \quad \forall\,\boldsymbol{w}^h\in V^h, \tag{33}$$

$$(\beta\nabla\varphi^h,\nabla\psi^h)_H-(\mathcal{E}\varepsilon(\boldsymbol{u}^h),\nabla\psi^h)_H+\int_{\Gamma_C}k_e\chi_{[0,\infty)}(u_\nu^h-g^h)(\varphi^h-\varphi_f^h)\psi^h\,da = (q^h,\psi^h)_W \quad \forall\psi^h\in W^h. \tag{34}$$

In a similar way, the discrete version of the regularized Problem $P_{V\,\delta}$ can be formulated as follows.
Problem $P_{V\,\delta}^h$. Find a discrete displacement $\boldsymbol{u}_\delta^h \in V^{\boldsymbol{h}}$ and a discrete electric potential $\varphi_\delta^h \in \mathrm{W}^{\boldsymbol{h}}$ such that

$$\left(\mathcal{F}\varepsilon(\boldsymbol{u}_\delta^h),\varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}}+\left(\mathcal{E}^T\nabla\varphi_\delta^h,\varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}}+\int_{\Gamma_C}c_\nu\left(u_{\delta_\nu}^h-g^h\right)_+w_\nu^h\,da = (\boldsymbol{f}^h,\boldsymbol{w}^h)_V \quad \forall\,\boldsymbol{w}^h\in V^h, \tag{35}$$

$$\left(\beta\nabla\varphi_\delta^h,\nabla\psi^h\right)_H-\left(\mathcal{E}\varepsilon(\boldsymbol{u}_\delta^h),\nabla\psi^h\right)_H+\int_{\Gamma_C}k_e\psi_\delta(u_{\delta_\nu}^h-g^h)(\varphi_\delta^h-\varphi_f^h)\psi^h\,da = (q^h,\psi^h)_W \quad \forall\psi^h\in W^h. \tag{36}$$

We now describe the numerical solution of the variational Problems P_V^h and $P_{V\,\delta}^h$. The numerical treatment of the condition of normal compliance is based on the penalty approach (see [8-10] for more details). Let N_{tot}^h be the total number of nodes and denote by α^i, β^i the basis functions of the spaces $V^{\boldsymbol{h}}$ and $W^{\boldsymbol{h}}$, respectively, for $i=1,\dots,N_{tot}^h$. Then, the expression of functions $\boldsymbol{w}^h \in V^{\boldsymbol{h}}$ and $\psi^h \in W^{\boldsymbol{h}}$ is given by $\boldsymbol{w}^h=\sum_{i=1}^{N_{tot}^h}\boldsymbol{w}^i\alpha^i$, $\psi^h=\sum_{i=1}^{N_{tot}^h}\psi^i\beta^i$ where $\boldsymbol{w}^i$ and ψ^i represent the values of the corresponding functions $\boldsymbol{w}$ and ψ at the i^{th} node of the uniform triangulation of Ω, denoted by T^h.

The penalized approach we use shows that the Problem P_V^h can be governed by the system of non-linear equations

$$R(\boldsymbol{u},\varphi)=\mathrm{G}(\boldsymbol{u},\varphi)+F(\boldsymbol{u},\varphi)=\boldsymbol{0}, \tag{37}$$

where the functions G and F are defined below. Here, the vectors $\boldsymbol{u}\in\mathbb{R}^{d\times N_{tot}^h}$ and $\varphi\in\mathbb{R}^{N_{tot}^h}$ are defined by $\boldsymbol{u}=\{\boldsymbol{u}^i\}_{i=1}^{N_{tot}^h}$ and $\varphi=\{\varphi^i\}_{i=1}^{N_{tot}^h}$, where $\boldsymbol{u}^i$ and φ^i represent the value of the function $\boldsymbol{u}^h$ and φ^h at the i^{th} nodes of T^h.
Next, the electro-elastic term $G(\boldsymbol{u},\varphi)\in\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}$ is defined by

$$\begin{aligned}\left(G(\boldsymbol{u},\varphi)\cdot(\boldsymbol{w},\psi)\right)_{\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}} &= \left(\mathcal{F}\varepsilon(\boldsymbol{u}^h),\varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}}+\left(\mathcal{E}^T\nabla\varphi^h,\varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}}-(\boldsymbol{f}^h,\boldsymbol{w}^h)_V+(\beta\nabla\varphi^h,\nabla\psi^h)_H \\ &\quad -(\mathcal{E}\varepsilon(\boldsymbol{u}^h),\nabla\psi^h)_H-(q^h,\psi^h)_W \\ &\quad \forall\,\boldsymbol{w}\in\mathbb{R}^{d\times N_{tot}^h},\quad \forall\,\psi\in\mathbb{R}^{N_{tot}^h},\quad \forall\,\boldsymbol{w}^h\in V^h,\quad \forall\psi^h\in W^h.\end{aligned} \tag{38}$$

Above, $\boldsymbol{w}$ and ψ represent the vectors of components $\boldsymbol{w}^i$ and ψ^i, for $i=1,\dots,N_{tot}^h$ respectively. Finally, the electro-mechanical contact operator $F(\boldsymbol{u},\varphi)\in\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}$ is defined by

$$\begin{aligned}\left(F(\boldsymbol{u},\varphi),(\boldsymbol{w},\psi)\right)_{\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}} &= \int_{\Gamma_C}c_\nu(u_\nu^h-g^h)_+w_\nu^h\,da+\int_{\Gamma_C}k_e\chi_{[0,\infty)}(u_\nu^h-g^h)(\varphi^h-\varphi_f^h)\psi^h\,da \\ &\quad \forall\,\boldsymbol{w}\in\mathbb{R}^{d\times N_{tot}^h},\forall\,\psi\in\mathbb{R}^{N_{tot}^h}.\end{aligned} \tag{39}$$

Note that, in the case of regularized Problem $P_{V\,\delta}^h$, the contact operator $F(\boldsymbol{u},\varphi)\in\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}$ is given by

$$\left(F(\boldsymbol{u},\varphi),(\boldsymbol{w},\psi)\right)_{\mathbb{R}^{d\times N^h_{tot}}\times\mathbb{R}^{N^h_{tot}}} = \int_{\Gamma_C} c_\nu (u^h_\nu - g^h)_+ w^h_\nu \, da \; + \int_{\Gamma_C} k_e \psi_\delta (u^h_\nu - g^h)(\varphi^h - \varphi^h_f)\psi^h \, da \tag{40}$$

$$\forall\, \boldsymbol{w} \in \mathbb{R}^{d\times N^h_{tot}}, \forall\, \psi \in \mathbb{R}^{N^h_{tot}}.$$

The solution of the non-linear system (37) is based on a linear iterative method similar to that used in the Newton method, which permits to treat simultaneously the two unknowns $\boldsymbol{u}$ and φ and, for this reason, we use in what follows the notation $\mathbf{x} = (\boldsymbol{u},\varphi)$. This Newton algorithm can be summarized by the following iteration process

$$\mathbf{x}^{\mathrm{i+1}} = \mathbf{x}^{\mathrm{i}} - (K^i + T^i)^{-1} R(\boldsymbol{u}^i, \varphi^i), \tag{41}$$

where $\mathbf{x}^{\mathrm{i+1}}$ denotes the pair $(\boldsymbol{u}^{i+1},\varphi^{i+1})$ and i represent the Newton iteration index; $K^i = D_{\boldsymbol{u},\varphi} G(\boldsymbol{u}^i,\varphi^i)$ represents the elastic matrix and $T^i = D_{\boldsymbol{u},\varphi} F(\boldsymbol{u}^i,\varphi^i)$ is the contact tangent matrix; also, $D_{\boldsymbol{u},\varphi} G$ and $D_{\boldsymbol{u},\varphi} F$ denote the differentials of the functions G and F with respect to the variables $\boldsymbol{u}$ and φ. This leads us to solve the resulting linear system

$$(K^i + T^i)\Delta\mathbf{x}^i = -R(\boldsymbol{u}^i,\varphi^i), \tag{42}$$

where $\Delta\mathbf{x}^i = (\Delta\boldsymbol{u}^i, \Delta\varphi^i)$ with $\Delta\boldsymbol{u}^i = \boldsymbol{u}^{i+1} - \boldsymbol{u}^i$ and $\Delta\varphi^i = \varphi^{i+1} - \varphi^i$.

Note that formulation (37) has been implemented in the open-source finite element library GetFEM++ (see [11]).

4. NUMERICAL SIMULATIONS

For the numerical simulations, we consider the physical setting depicted in Figure 1. In this case, the body $\Omega = (0\,,\ 3) \times (0\,,1) \subset \mathbb{R}^2$ is clamped on $\Gamma_D = \{0\} \times [0\,,1] \cup \{3\} \times [0\,,1] = \Gamma_b$. Vertical tractions act on $\Gamma_N = [0\,,1] \times \{3\}$, i.e., $\boldsymbol{f}_N = (0,-60) N/m$ and the electric potential is free there (we choose $\Gamma_N = \Gamma_a$). The body is in contact with a conductive foundation on its lower boundary $\Gamma_C = [0\,,3] \times \{0\}$. No volume forces and no electric charges are supposed to act in the body, i.e. $\boldsymbol{f}_0 = \mathbf{0}\, N/m^2,\ q_0 = 0\, C/m^2$ and $q_b = 0\, C/m$.

In the plane of deformations setting, the constitutive law in Equations (1) and (2) can be written by using a compressed matrix notation in place of the tensor notation as follows

$$\begin{bmatrix}\sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ D_2\\ D_3\end{bmatrix} = \begin{bmatrix} f_{22} & f_{23} & 0 & 0 & e_{32}\\ f_{23} & f_{33} & 0 & 0 & e_{33}\\ 0 & 0 & f_{44} & e_{24} & 0\\ 0 & 0 & e_{24} & -\beta_{22} & 0\\ e_{32} & e_{33} & 0 & 0 & -\beta_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ -E_2\\ -E_3\end{bmatrix}.$$

Here, we use as material the piezoelectric body whose constants are taken as [1]:

- Elastic $[GPa]$: $f_{22} = 210$, $f_{23} = 105$, $f_{33} = 211$, $f_{44} = 42.5$;
- Piezoelectric $[C/m^2]$: $e_{32} = -0.61$, $e_{33} = 1.14$, $e_{24} = -0.59$;
- Permittivity $[C/GVm]$: $\beta_{22} = -0.073$, $\beta_{33} = -0.077$.

The following data have been used in the numerical simulations:

$$c_\nu = 10^7\, N/m^2, \qquad g = 0.1\, m, \qquad k_e = 1, \qquad \delta = 10^{-7} m, \qquad \varphi_f = -64\, V.$$

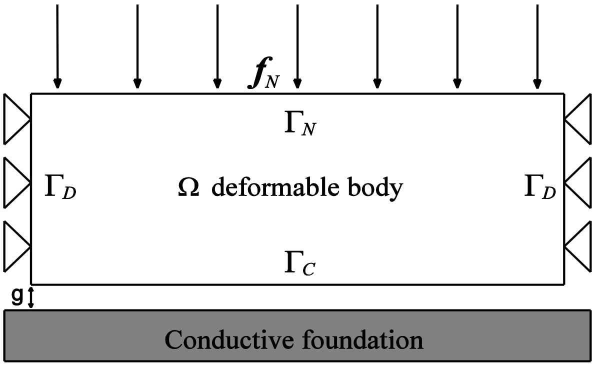

Figure 1. Physical setting

The deformed configuration of the body is represented in Figure 2 (left), which corresponds to the numerical solution of problem P_V. In order to compare the deformed mesh related to Problem P_V with that obtained for the numerical solution of Problem P_{V_δ}, we plotted in Figure 2 (right) the deformed configuration for the numerical solution of the regularized problem P_{V_δ}. The numerical solution presented in figure 2 corresponds to a meshsize $h = 1/16$: the spatial domain is discretized into 1536 triangular elements with 48 contact elements.

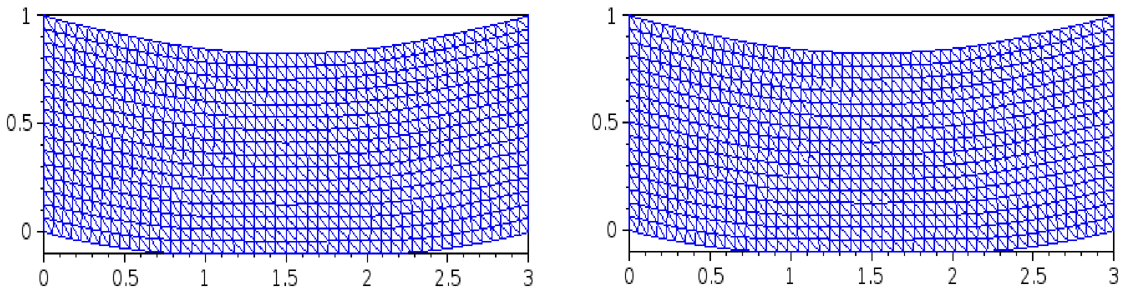

Figure 2. Amplified deformed mesh: the original contact problem (left) and the regularized problem (right)

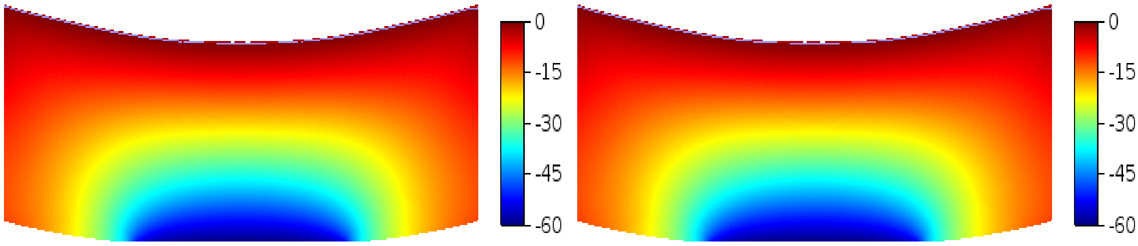

Figure 3. Electric potential: the original contact problem (left) and the regularized problem (right)

In Figure 3, the electric potential is plotted on the deformed configuration. These simulations describe the inverse piezoelectric effect, i.e. the appearance of strain in the body, due to the action of the electric field. Also, they underline the effects of the electrical conductivity of the foundation on the process. The reverse effect is used in actuators: a piezoelectric actuator can be produced to protect materials already in contact; this in order not to exceed the voltage thresholds specified by the manufacturer. According to the Figures 2 and 3, we observe that the numerical results obtained for the solution of Problem P_V are very well approximated by the solution of Problem P_{V_δ}. Next, we lead a parametric study according to the regularized coefficient δ. To this end, in Figure 4 we study the convergence on the whole discrete domain Ω^h of the displacement and the electric potential solutions obtained for Problem P_{V_δ} towards that obtained for Problem P_V. Here, we consider the numerical estimation of the difference $||\boldsymbol{u}_\delta^h - \boldsymbol{u}^h|| + ||\varphi_\delta^h - \varphi^h||$ between the numerical solutions obtained for Problems P_V and P_{V_δ}. The results depicted in Figure 4 illustrate that the solution of the regularized problem gives a reliable and accurate approximation of the original problem, provided that the regularization parameter takes very small values.

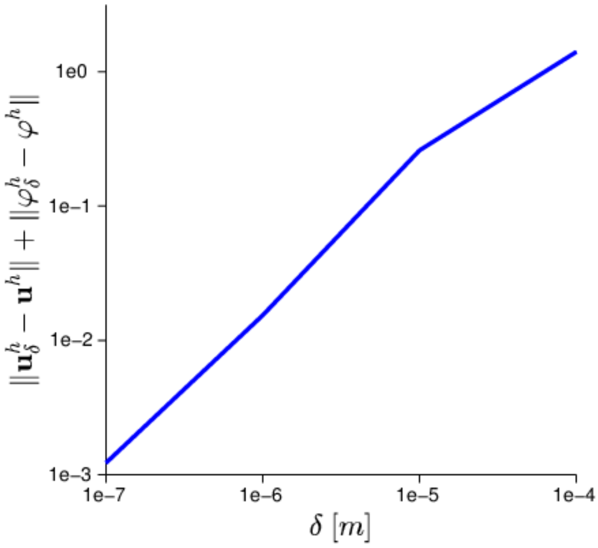

Figure 4. Convergence result

5. CONCLUSION

In this paper, piezoelectric contact is numerically studied. The novelties arise in the fact that an electro-elastic constitutive law describes the material behavior and the foundation is electrically conductive. A discrete scheme was used to approach the problem and a numerical algorithm, which combines the penalized approach with the Newton method, was implemented. Moreover, numerical simulation results are reported on a two-dimensional test problem. These simulations provide a reliable comparison between numerical solutions of the regularized problem and the original one. In addition to the mathematical interest in the convergence result shown in Figure 4, it is important from the mechanical point of view since is shows that the solution of the contact problem can be approached by the solution of a regularized contact problem as the regularization parameter converges to zero. Performing these simulations, we found that the numerical solution worked well and the convergence was rapid. This work opens the way to study further models of frictional contact with other conditions for thermally-electrically conductive foundation taking into the account the heat exchange condition.

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