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Hardy's theorem for the generalized Bessel transform on the half line

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ABSTRACT

In this paper, we give a generalization of a qualitative uncertainty principle namely Hardy's theorem, which asserts that a function and its Fourier transform cannot both be very small, for the generalized Bessel transform on the half line.

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1. INTRODUCTION

The uncertainty principle says that a function and its Fourier transform can't simultaneously decay very rapidly at infinity. A classical version of uncertainty principle, known as Hardy's theorem, was first proved by Hardy on $\mathbb{R}$. We state Hardy's theorem on $\mathbb{R}$ as follows [3].

Theorem 1..1 *Suppose that f is a measurable function on* $\mathbb{R}$ *and satisfies*

$$|f(x)| \leq Ce^{-ax^2}$$

$$|F(f)(\xi)| \leq Ce^{\frac{-\xi^2}{4a^2}}$$

then f is a multiple of e^{-ax^2}

The Hardy's theorem was extended to various settings see [4–6] for more results. The purpose of this paper is to obtain a generalization of this theorem for the generalized Bessel transform.

The structure of the paper is as follows: In section 2 we set some notations and collect some basic results about the Bessel operator and the Bessel transform. In section 3 we give some facts about harmonic analysis related to the second-order singular differential operator on the half line Δ and generalized Bessel transform. In section 4 we state and prove an analogue of Hardy's theorem for the generalized Bessel transform.

2. PRELIMINARIES

In this section, we recapitulate some facts about harmonic analysis related to the Bessel operator $\mathcal{L}_\alpha$. We cite here, as briefly as possible, some properties. For more details we refer to [7].
Throughout this paper we assume that $\alpha > \frac{-1}{2}$.
Defined L^p_α, $1 \leq p \leq \infty$, as the class of measurable function f on $[0, +\infty[$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{\frac{1}{p}} \quad if\ p < \infty$$

and

$$\|f\|_{\infty,\alpha} = \|f\|_\infty = ess\ sup_{x\geq 0}|f(x)|.$$

The Bessel operator $\mathcal{L}_\alpha$ is defined as following:

$$\mathcal{L}_\alpha f(x) = \frac{d^2}{dx^2} f(x) + \frac{2\alpha+1}{x}\frac{d}{dx} f(x).$$

The Fourier-Bessel transform of ordre α is defined for a function $f \in L^1_\alpha$ by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx, \quad \lambda \geq 0, \tag{1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^\infty \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\,\Gamma(n+\alpha+1)} \ (z \in \mathbb{C}). \tag{2}$$

is the normalized Bessel function of index α.

Proposition 2..1 **(i)** *If both f and $\mathcal{F}_\alpha$ are in L^1_α then*

$$f(x) = \int_0^\infty \mathcal{F}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda), \quad for\ almost\ all\ x \geq 0$$

where

$$d\mu_\alpha(\lambda) = \frac{1}{4^\alpha(\Gamma(\alpha+1))^2}\lambda^{2\alpha+1} d\lambda. \tag{3}$$

(ii) *For every $f \in L^1_\alpha \bigcap L^2_\alpha$ we have*

$$\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx = \int_0^\infty |\mathcal{F}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

The Bessel translation operators τ^x_α, $x \geq 0$, are defined by

$$\tau^x_\alpha(f)(y) = a_\alpha \int_0^\pi f(\sqrt{x^2+y^2+2xycos\theta})(sin\theta)^{2\alpha} d\theta, \tag{4}$$

where

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}. \tag{5}$$

3. HARMONIC ANALYSIS ASSOCIATED WITH Δ

In this section we provide some facts about harmonic analysis related to the second-order singular differential operator on the half line Δ. We cite here, as briefly as possible, some properties. For more details we refer to [1, 2].
Consider the second-order singular differential operator on the half line

$$\Delta f(x) = \frac{d^2}{dx^2} f(x) + \frac{2\alpha+1}{x}\frac{d}{dx} f(x) - \frac{4n(\alpha+n)}{x^2} f(x)$$

where $\alpha > \frac{-1}{2}$ and $n = 0, 1, 2, ...$ For $n = 0$ we regain the Bessel operator $\mathcal{L}_\alpha$.
Let $\mathcal{M}$ be the map defined by

$$\mathcal{M}f(x) = x^{2n}f(x).$$

Let $L^p_{\alpha,n}, 1 \leq p \leq \infty$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \|\mathcal{M}^{-1}f\|_{p,\alpha+2n} < \infty.$$

Remark 3..1 *$\mathcal{M}$ is an isometry from $L^p_{\alpha+2n}$ onto $L^p_{\alpha,n}$.*

3.1. Generalized Bessel transform

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n}j_{\alpha+2n}(\lambda x), \quad (6)$$

where $j_{\alpha+2n}$ is the normalized Bessel function of index $\alpha + 2n$ given by (2).

Proposition 3..1

- *φ_λ possesses the Laplace integral representation*

$$\varphi_\lambda(x) = a_{\alpha+2n}x^{2n}\int_0^1 cos(\lambda tx)(1-t^2)^{\alpha+2n-\frac{1}{2}}dt, \quad (7)$$

where $a_{\alpha+2n}$ is given by (5)

- *φ_λ satisfies the differential equation*

$$\Delta\varphi_\lambda = -\lambda^2\varphi_\lambda$$

- *For all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$,*

$$|\varphi_\lambda(x)| \leq x^{2n}e^{|Im\lambda||x|}.$$

Definition 3..2 *The generalized Fourier transform is defined for a function $f \in L^1_{\alpha,n}$ by*

$$\mathcal{F}_\Delta(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx, \quad \lambda \geq 0. \quad (8)$$

Remark 3..2

- *By (1) and (3) observe that*

$$\mathcal{F}_\Delta = \mathcal{F}_{\alpha+2n} \circ \mathcal{M}^{-1}, \quad (9)$$

where $\mathcal{F}_{\alpha+2n}$ is the Fourier-Bessel transform of order $\alpha + 2n$ given by (1).

- *If $f \in L^1_{\alpha,n}$ then $\mathcal{F}_\Delta(f) \in C_0([0;\infty[)$(of continuous functions on $[0;\infty[$ vanishing at infinity) and $\|\mathcal{F}_\Delta(f)\|_\infty \leq \|f\|_{1,\alpha,n}$.*

Theorem 3..3 *Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Delta(f) \in L^1_{\alpha+2n}$. Then for almost all $x \geq 0$,*

$$f(x) = \int_0^\infty \mathcal{F}_\Delta(f)(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where $d\mu_{\alpha+2n}(\lambda)$ is given by (3).

Theorem 3..4 **(i)** *For every $f \in L^1_{\alpha,n} \bigcap L^2_{\alpha,n}$ we have the Plancherel formula*

$$\int_0^\infty | f(x) |^2 x^{2\alpha+1}dx = \int_0^\infty | \mathcal{F}_\Delta(f)(\lambda) |^2 d\mu_{\alpha+2n}(\lambda).$$

(ii) *The generalized Fourier transform $\mathcal{F}_\Delta$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,\infty[,\mu_{\alpha+2n})$. The inverse transform is given by*

$$\mathcal{F}^{-1}_\Delta(g)(x) = \int_0^\infty g(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where the integral converge in $L^2_{\alpha,n}$.

4. HARDY'S THEOREM FOR THE GENERALIZED BESSEL TRANSFORM

In this section, we will obtain a Hardy uncertainty principle for the generalized Bessel transform.

Theorem 4..1 *Suppose that f is a measurable function such that $f \in L^2_{\alpha,n}$ and satisfies*

$$|f(x)| \leq Cx^{2n}(1+x^2)^k e^{-ax^2}$$

$$|\mathcal{F}_\Delta(f)(\xi)| \leq C(1+\xi^2)^k e^{-b\xi^2}$$

where $a, b > 0$. Then, $f = 0$ whenever $ab > \frac{1}{4}$ and when $ab = \frac{1}{4}$, $f(x) = H(x)e^{-ax^2}$, where H is a polynomial of degree $\leq 2k + 2n$.

We will use the following lemma

Lemma 4..2 *[8] Suppose that $F(\xi)$ is an entire function of one complex variable satisfying*

$$|F(\xi)| \leq C(1+|\xi|^2)^k e^{b|Img(\xi)|^2}, \quad \xi \in \mathbb{C}$$

$$|F(\xi)| \leq C(1+\xi^2)^k e^{-b\xi^2}, \quad \xi \in \mathbb{R}$$

where b is a positive constant. Then, $F(\xi) = P(\xi)e^{-b\xi^2}$, where $P(\xi)$ is a polynomial of degree $\leq 2k$.

Proof of theorem

Assume first that $ab = \frac{1}{4}$. Obviously, $\mathcal{F}_\Delta(f)$ can be extended to an entire function. Let $\xi = \zeta + i\eta$ then, we have

$$\begin{aligned}
|\mathcal{F}_\Delta(f)(\xi)| &= C|\int_0^\infty f(x)\varphi_\xi(x)x^{2\alpha+1}dx| \\
&= C|\int_0^\infty x^{2n}f(x)j_{\alpha+2n}(\xi x)x^{2\alpha+1}dx| \\
&\leq \frac{C}{2}|\int_0^\infty \frac{f(x)}{x^{2n}}x^{2(\alpha+2n)+1}\int_{-1}^1 e^{ix\xi s - x\eta s}(1-s^2)^{\alpha+2n-\frac{1}{2}}dsdx| \\
&\leq \frac{C}{2}|\int_0^\infty (1+x^2)^k x^{2(\alpha+2n)+1}e^{-ax^2+x|\eta|}dx| \\
&\leq \frac{C}{2}e^{b\eta^2}\int_0^\infty (1+x^2)^k x^{2(\alpha+2n)+1}e^{-(\sqrt{a}x-\sqrt{b}|\eta|)^2}dx \\
&\leq \frac{C}{2}(1+\eta^2)^{k+\alpha+2n+1}e^{b\eta^2} \\
&\leq C(1+\eta^2)^{k+\alpha+2n+1}e^{b\eta^2}.
\end{aligned}$$

By lemma 4.2, $\mathcal{F}_\Delta(f)(\xi) = Q(\xi)e^{-b\xi^2}$, where $Q(\xi)$ is a polynomial. Because

$$|Q(\xi)e^{-b\xi^2}| = |\mathcal{F}_\Delta(f)(\xi)| \leq C(1+\xi^2)^k e^{-b\xi^2}$$

we have $degQ = 2k$.

In view of Lemma 4.2 and by taking the inverse of the Fourier-Bessel transform of order $\alpha + 2n$, we obtain

$$\mathcal{M}^{-1}(f)(x) = P(x)e^{-ax^2},$$

where $P(x)$ is an even polynomial of degree $\leq 2k$.
Then

$$f(x) = x^{2n}P(x)e^{-ax^2} = H(x)e^{-ax^2}$$

where $H(x)$ is an even polynomial of degree $\leq 2k + 2n$.
When $ab > \frac{1}{4}$, $|\mathcal{M}^{-1}(f)(x)| \leq C(1+|x|^2)^k e^{-a_1x^2}$, where $a_1 = \frac{1}{4b} < a$.
As argument above,

$$\mathcal{M}^{-1}f(x) = P(x)e^{-a_1x^2}$$

where $P(x)$ is an even polynomial of degree $\leq 2k$.
Then

$$|\mathcal{M}^{-1}f(x)| \leq C(1+x^2)^k e^{-ax^2}$$

cannot hold unless

$$\mathcal{M}^{-1}f = 0,$$

then $f = 0$.

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