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Quantum Feature Of Branched Hamiltonians

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ABSTRACT

We pointed out that a quadratic Liénard-type equation, when appropriately interpreted, exhibited branching behavior as a consequence of the double-valued nature of its governing Hamiltonian. Under a suitable approximation involving the coupling constant, we derived the corresponding quantum mechanical model, characterized by a momentum-dependent effective mass function.

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1 Introduction

The exploration of non-conventional Hamiltonians, particularly those exhibiting($(\mathcal{PJ})$ symmetric schemes [1], has garnered significant interest due to their applicability in various domains such as nonlinear dynamics, shallow water-wave phenomena, and quantum mechanical models, including their relativistic extensions [2]. Branched Hamiltonians [3, 4, 5] represent a fascinating and non-conventional area within mathematical physics, arising from Lagrangians with non-convex velocity dependencies. These systems lead to multi-valued Hamiltonians, introducing complex phasespace structures and topological considerations. Branched Hamiltonians, characterized by their multivalued dependence on momentum, present intriguing challenges, and opportunities in both classical and quantum mechanics. These systems arise from Lagrangians with non-convex velocity dependencies, leading to complex phasespace structures and necessitating a revaluation of traditional quantization, have been recently proposed by Shapere and Wilczek [3, 4, 5] and is currently an area of active interest as evidenced in a series of papers by Curtright and Zachos [6, 7, 8, 9, 10, 11] and other works [12, 13, 14]. It ought to be mentioned that in a broader methodical framework the traditional approaches often do not work that well: indeed, their applicability is challenged by various innovative theoretical motivations. Just to name a few, the growth of popularity of the detailed study of certain non-real, complex potentials with real spectra or, most recently, of their friendly and tractable non-local generalizations [15].
An interesting trend has emerged that addresses different types of non-linear interactions which are mediated, say, by a set of dynamical parameters that are energy dependent [16]. This could be carried out by the dynamical interpaly of the role from the guiding potential of the system to for instance varying mass characterization [17] etc. Recent works have especially focused attention [5] to elementary classical models and succeeded, in some

cases, in the construction of their quantum counterparts. In this context we can speak of partner Hamiltonians of supersymmetric quantum mechanics paired by supercharges inducing an isospectral system in a momentum space framework. Although the potential appears in the Lagrangian in the conventional subtracted way, when one switches to the momentum space one inevitably encounters the harmonic oscillator. We have explored the possibility of modifications by connecting to nonlinear models that admit a Hamiltonian representation. In regard we should mention that local branching is not so sufficient to ensure integrability. In fact, an integrable differential equation having solutions that are not locally finitely branched with a finitely sheeted Riemann surface but not yet identified through Painlevé analysis, is in itself an interesting open problem [7].

In the present work, we have searched for situations where multi-valued Hamiltonians occur that result from the enforcement of the Legendre transform on the non-standard Lagrangians in the sense that the velocity dependence is not convex. Early works by Curtright and Zachos [7, 8, 9, 10, 11, 12] and by Bagchi et al. [18] pointed out that multi-valued Hamiltonians arose in the continuous interpolation of discrete time dynamical systems that invariable had an underlying chaotic behaviour. The natural framework of study would then be a quasi-Hamiltonian formalism. In our approach we constructed systems where a simple generic Lagrangian toy model led to multiple-valued Hamiltonian systems. It was observed that such Hamiltonians would almost always be defined in the momentum space revealing systems with a momentum-dependent mass function. The underlying Hamiltonians show interesting supersymmetric analogies. We also expect that such systems would exhibit chaotic behaviour thus making the connection to dynamical systems very transparent. In the context of nonlinear systems, the Liénard class presents an intriguing feature of the Hamiltonian in that it depicts a system in which the roles of the position and momentum variables are exchanged. In [8] two eligible elementary nonlinear models were proposed which admitted a feasible quantization. Liénard equation represents a cubic oscillator subject to a damped nonlinear force. Naturally, the presence of the damping poses to be a problem whenever one tries to contemplate a quantization of the model. Liénard systems are of potential importance not only in optics [9] but also in the shallow water-wave studies [10] and in non-Hermitian quantum mechanics [13]. Due to nonlinearity one of the main obstacles lies in the absence of superposition principle. However, it is important to realize that the quantization is hard to tackle in the coordinate representation of the Schrödinger equation but can be straightforwardly carried out in the momentum space. Such a Hamiltonian, which has the feature of being a branched type, may be interpreted to represent a momentum-dependent effective mass quantum system. While the complementary quantum problem of the position-dependent mass has received considerable attention due to its relevance in describing the dynamics of electrons in problems of compositionally graded crystals, quantum dots and liquid crystals, interest in momentum-dependent mass problems is a somewhat recent curiosity [13] arising from the observation that the PT- symmetric Liénard type nonlinear oscillator can afford complete solvability in a momentum space description. Bagchi and his collaborators [18] explored certain classes of branched Hamiltonians for the well-known class of nonlinear autonomous differential equations that are of Liénard type [19, 20, 21].

2 Branched Hamiltonians: A brief review

Let us discuss the example of a branched system as illustrated in [7]. It is to be noted here that a typical classical model of the above-mentioned non-quadratic type may be sampled by Lagrangian

$$L = C(v-1)^{\frac{2k-1}{2k+1}} - V(x), \text{ where } C = \frac{2k+1}{2k-1}\left(\frac{1}{4}\right)^{\frac{2}{2k+1}} \tag{1}$$

Function $V(x)$ represents a convenient local interaction potential while the traditional kinetic-energy term is tentatively replaced by a fairly unusual function of "velocity" v.

This model was recently analyzed in Ref. [7] where the definition of the fractional powers of difference $v-1$ was adapted to the needs of possible phenomenology. In detail, the $(2k+1)$-st root was required real and positive or negative for $v>1$ or $v<1$, respectively. By doing this we are in fact taking the real parts of two different branches of the analytic $(2k+1)$-st root as a function of complex v. We do this solely to have a real, single-valued Lagrangian function for all real v.

Under the circumstances, there is every reason to draw on more than one branch of an analytic function of v provided that only one branch is encountered at any given real v, or at least that would seem to be true for classical dynamics. The following graph, will illustrate the consequences this choice for L has for the quantum dynamics is, especially for the case $k=1$.

If we expand the above Lagrangian for the case $k=1$ by Taylor's series for v near zero, we then have $L \approx C\left(-1+\frac{v}{3}+\frac{v^2}{9}+O(v^3)\right)-V(x)$. Of these terms, the first

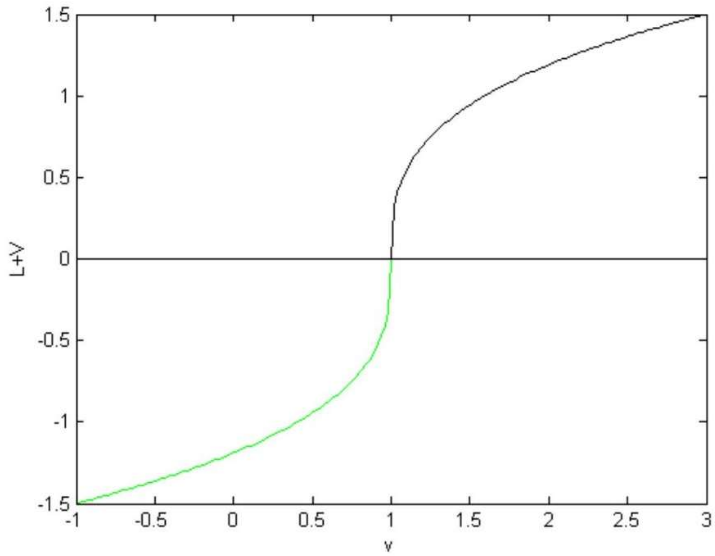

Figure 1: $L + V = C(v-1)^{\frac{1}{3}}$ for the case $k = 1$.

is naive, the second would give a boundary contribution to the action and therefore not effect the equations of motion, and the third is the usual v^2 kinetic structure:

$$A = \int_{t_1}^{t_2} L dt$$

$$\approx C(t_2 - t_1 + \frac{1}{3}\left(x(t_2) - x(t_1)\right) + \frac{1}{9}\int_{t_1}^{t_2} \mathrm{v}^2 dt + \int_{t_1}^{t_2} \mathrm{O}(\mathrm{v}^3) dt) - \int_{t_1}^{t_2} V(x) dx \quad (2)$$

So, this action would result in the usual Newtonian classical equations of motion for small v. On the other hand, for large velocities, the v dependence is more elaborate, leading (for finite, positive integer k) to a non-convex function of velocity, whose curvature $\partial^2 L/\partial v^2$ flips sign at just one point, namely, $v = 1$.
Thus, the function L may be thought of a single pair of convex functions judiciously pieced together. The non-convexity of L has the effect of making the Kinetic energy, and hence the Hamiltonian, a double-valued function of p.
In the case of the Lagrangian (1), the canonical momentum turns out to be

$$p = \left(\frac{1}{4}\right)^{\frac{2}{2k+1}} \frac{1}{(v-1)^{\frac{2}{2k+1}}} \quad (3)$$

whose inversion immediately tells us that the velocity variable $v(p)$ is a double-valued function of p.

$$v_{\pm}(p) = 1 \mp \frac{1}{4}\left(\frac{1}{\sqrt{p}}\right)^{(2k+1)} \quad (4)$$

This has the implication that if we evaluate the Hamiltonian its branches $H_{\pm}$will appear and hence the Hamiltonian, a double-valued function of p. For any positive integer k, we find two branches for H,

$$H_{\pm} = p \pm \frac{1}{4k-2}\left(\frac{1}{\sqrt{p}}\right)^{2k-1} + V(x) \quad (5)$$

For $k = 1$, the two kinetic energy branches have the shape shown in the figure below. Note that, classically, p must be non-negative for this model to avoid imaginary $v(p)$. That is to say, the slope $\frac{\partial L}{\partial v}$ is always positive.

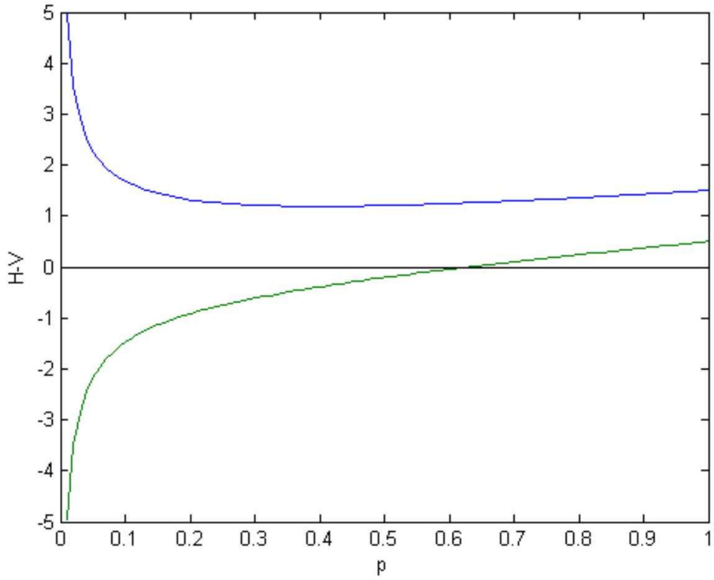

Figure 2: $H_{\pm} - V(x) = p \pm \frac{1}{2\sqrt{p}}$ in blue/green. There is a cusp at $p = \infty$.

We also note that $k = 1$ case speaks of the canonical supersymmetric structure for the difference $H_{\pm} - V(x)$ namely, $p \pm \frac{1}{2\sqrt{p}}$, but in the momentum space if viewed as a quantum mechanical system. The spectral and boundary condition linkages of these Hamiltonian are not difficult to set up.

3 Branched Hamiltonians for Liénard type systems

Initial efforts to identify branched Hamiltonians within the framework of nonlinear differential equations focused on the linear subclass of models of the type Liénard [9]. In this work, we extend the analysis to explore the presence of branched Hamiltonians in a broader class of nonlinear, autonomous Liénard-type differential equations. Such systems are of considerable relevance in various physical contexts, including nonlinear optics, shallow water wave phenomena, and non-Hermitian quantum mechanics. A significant challenge posed by the inherent nonlinearity of these equations is the breakdown of the superposition principle, which complicates analytical treatment and solution construction.

The class of Liénard equations is represented by

$$\ddot{x} + g(x)\dot{x} + h(x) = 0, \tag{6}$$

where an overdot indicates a derivative with respect to the time (t) variable while $g(x)$ and $h(x)$ are two continuously differentiable functions of the spatial coordinate x.

We concentrate on the model of Mathews and Lakshmanan [21] which looks at the case wherein $g(x) = kx$ and $h(x) = \lambda x + \frac{k^2}{9}x^3$ (these are odd functions of x) and which yields the equation of motion

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda x = 0, k\lambda > 0. \tag{7}$$

representing a cubic oscillator subject to a damped nonlinear force as indicated by the product term $kx\dot{x}$. This equation is appealing in that it can be seen to follow from a Lagrangian as provided by the precise form

$$L = \frac{27\lambda^3}{2k^2}\left(k\dot{x} + \frac{k^2x^2}{3} + 3\lambda\right)^{-1} + \frac{3\lambda\dot{x}}{2k} - \frac{9\lambda^2}{2k^2}. \tag{8}$$

After some straightforward manipulations the corresponding Hamiltonian H can also be written down

$$H_{(x,p)} = \frac{9\lambda^2}{2k^2}\left[2 - 2\left(1 - \frac{2kp}{3\lambda}\right)^{1/2} + \frac{k^2x^2}{9\lambda} - \frac{2kp}{3\lambda} - \frac{2k^3x^2p}{27\lambda^2}\right]. \tag{9}$$

One can see that H is of non-standard type: the coordinate and momentum are mixed so that the Hamiltonian cannot be written as the sum of individual kinetic and potential energy terms.

The Lagrangian (8), however, is not unique in producing the Hamiltonian $H_{(x,p)}$ as was pointed out in [9]. Towards this end Bagchi et al. [9] considered an entirely different form

$$L(x,v) = C(v+f(x))^{\frac{2m+1}{2m-1}} - \delta, \text{ where } C = \frac{1-2m}{1+2m}\delta^{\frac{2}{1-2m}} \tag{10}$$

where v is the velocity variable, $f(x)$ is an arbitrary smooth function, m is a nonnegative integer and δ is some suitable constant quantity. The motivation for such a form came from the model of Curtright and Zachos [7] who were interested in seeking some concrete examples of branched Hamiltonians. The present one of (10) differs from it in the presence of additional distinguishing features in the form of an inverse exponent in the first term and the inclusion of a general function term $f(x)$. Since L inevitably leads to multiple values for p, the resulting system possesses a branched structure

$$H_{\pm}(x,p) = (-p)f(x) - \frac{2\delta}{2m+1}(\pm\sqrt{-p})^{2m+1} + \delta \tag{11}$$

Quite interestingly, for the specific case of $m = 0, H_{\pm}$reduce to

$$H_{\pm} = (-p)f(x) \mp 2\delta\sqrt{-p} + \delta \tag{12}$$

When one defines $f(x)$ and δ as

$$f(x) = \frac{\lambda}{2}x^2 + \frac{9\lambda^2}{2k^2}, \delta = \frac{9\lambda^2}{2k^2} \tag{13}$$

then with the trivial shift $p \to \frac{2k}{3\lambda}p - 1$, we get

$$H_{\pm} = \frac{9\lambda^2}{2k^2}\left[2 \mp 2\left(1 - \frac{2kp}{3\lambda}\right)^{1/2} + \frac{k^2x^2}{9\lambda} - \frac{2kp}{3\lambda} - \frac{2k^3x^2p}{27\lambda^2}\right] \tag{14}$$

Evidently, the positive branch corresponds to the form of $H_{(x,p)}$ in (8) while both $H_{\pm}$reveal the presence of a linear harmonic potential in the limit $k \to 0$. The pair Hamiltonians above speak of branching in the momentum space as p deviates from the value $\frac{3\lambda}{2k}$.

4 Quantization

Quantization of the branched system (14) can be handled in a typical way as detailed in [22], i.e., by adopting a suitable ordering procedure and implementing appropriate boundary conditions. The Hamiltonian (14) is of non-standard type. The co-ordinates and potentials are mixed so that the expression cannot be split into individual kinetic and potential energy terms. However, we can write

$$H_{(x,p)} = \frac{1}{2}f(p)x^2 + U(p), \tag{15}$$

where

$$f(p) = \omega^2\left(1 - \frac{2kp}{3\omega^2}\right), U(p) = \frac{9\omega^4}{2k^2}\left(\sqrt{1 - \frac{2kp}{3\omega^2}} - 1\right)^2, \tag{16}$$

with $\omega = \sqrt{\lambda}$. The roles of coordinate and momentum have been transposed implying a momentum-dependent system at play. The first term in (16) represents a mixed function of both position and momentum variables while the second term is a function of momentum alone. The classical Hamiltonian $H_{(x,p)}$ is invariant under a joint action of coordinate reflection and time reversal transformation. Exact quantization of the Hamiltonian can be carried out by going over to the momentum space with $\hat{x} = i\hbar\frac{\partial}{\partial p}$. The Hamiltonian turns out to be of momentumdependent mass type. Adopting a von Roos strategy of quantizing the problem by considering a general symmetric ordering [23], the underlying Schrodinger equation can be formulated and solved, in principle at least.

5 Summary

In this study, we examined a nonlinear oscillator model exhibiting quasi-harmonic oscillations and demonstrated that its corresponding Hamiltonian possesses a nonconventional, double-valued structure, arising from a velocity-dependent potential. For small values of the coupling parameter, we showed that the system admits a quantum mechanical interpretation, effectively describing a particle with a momentumdependent effective mass.

References

[1] C. M. Bender, P. E. Dorey, C. Dunning, A. Fring, D. W. Hook, H. F. Jones, S. Kuzhel, G. Levai, and R. Tateo, PJ Symmetry: In Quantum and Classical Physics, World Scientific, Singapore.
[2] B. P. Mandal, B. K. Mourya, K. Ali and A. Ghatak," PT phase transition in a (2+1)-d relativistic system", *Ann. Phys.* 363, 185 (2015).
[3] A. Shapere and F. Wilczek, "Branched Quantization", *Phys. Rev. Lett*. 109, 200402 (2012).
[4] A. Shapere and F. Wilczek," Classical time Crystals", *Phys. Rev. Lett.* 109, 160402 (2012).
[5] F. Wilczek," Quantum time Crystals" *Phys. Rev. Lett.* 109, 160401 (2012).
[6] T.L. Curtright, "Potentials unbounded Below ", *SIGMA* 7, 042 (2011).
[7] T. L. Curtright and C. K. Zachos," Branched Hamiltonians and Supersymmetry", J. *Phys. A: Math. Theor.* 47, 145201 (2014).
[8] T. L. Curtright, "The BASICSs of Branched Hamiltonians", *Bulg. J. Phys*. 45, 102 (2018).
[9] B. Bagchi, S. M. Kamil, T. R. Tummuru, I. Semoradova and M. Znojil, "Branched Hamiltonians for a class of velocity dependent potentials", *J. Phys.: Conf. Series* 839, 012011 (2017).
[10] A. Ghose-Choudhury and P. Guha, "Branched Hamiltonians and time translation symmetry breaking in equations of the Liénard type", *Mod. Phys. Lett. A* 34, 1950263 (2019).
[11] F. Bagarello, J. P. Gazeau, F. H. Szafraniec and M. Znojil, editors, (2015)" Non-Self adjoint Operators in Quantum Physics: Mathematical Aspects", John Wiley Sons.
[12] C. Quesne and V. M. Tkachuk, "Deformed algebras, position-dependent effective masses and curved spaces: an exactly solvable Coulomb problem", *J. Phys. A: Math. Gen*. 37, 4267 (2004).
[13] B. Bagchi, A. Banerjee, C. Quesne and V. M. Tkachuk, "Deformed shape invariance and exactly solvable Hamiltonians with position-dependent effective mass", *J. Phys. A: Math. Gen.* 38, 2929 (2005).
[14] B. Bagchi and T. Tanaka, "A generalized non-Hermitian oscillator Hamiltonian, N-fold supersymmetry and position-dependent mass models*" Phys. Lett. A* 372, 5390 (2008).
[15] T. L. Curtright, D. B. Fairlie, and C. K. Zachos, (2014) A Concise Treatise on Quantum Mechanics in Phase Space, Imperial College, and World Scientific Press; (ISBN 978-981-4520-43-0).
[16] B Bagchi, (2000) Supersymmetry in Quantum and Classical Mechanics, Chapman and Hall/CRC.
[17] D Ghosh "Nonstandard Lagrangians and a model of branched Hamiltonians" , *IOSR Journal of Applied Physics*.16, 21 (2024).
[18] B. Bagchi, D. Ghosh and T. R. Tummuru, "Branched Hamiltonians for a quadratic type Liénard oscillator*", J. Non. Evol. Eqns and Appl*. 2018, 101 (2020).
[19] D. Ghosh "A Generalized class of Branched Hamiltonians", *IOSR Journal of Applied Physics.* 17, 15 (2025).
[20] B. Bagchi, R. Ghosh and P. Goswami, "Generalized Uncertainty Principle and Momentum-Dependent Effective Mass Schrödinger Equation", *J. Phys: Conf. Ser.* 1540, 012004 (2020).
[21] M. Lakshmanan and S. Rajasekar, (2013) Nonlinear dynamics: Integrability chaos and patterns, Springer-Verlag, New York.
[22] V. Chithiika Ruby, M. Senthivelan and M. Lakshmanan, "Quantum solvability of a nonlinear δ-type mass profile system:coupling constant quantization", *J. Phys. A: Math. Theor.* 45, 382002 (2012).
[23] O. von Roos, "Position dependent effective masses in semiconductor theory", *Phys. Rev. B* 27, 7547 (1983).