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Stability analysis of double-diffusive convection in couple-stress Hall fluid

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ABSTRACT

The aim of the present work was to study the effects of uniform horizontal magnetic field and Hall currents on the double-diffusive convection in couple-stress fluid through permeable media. Following the linearized stability theory, Boussinesq approximation and normal mode analysis, the dispersion relation is obtained. The stationary convection, stability of the system and oscillatory modes are discussed. For the case of stationary convection, the stable solute gradient and magnetic field postpones the onset of convection while the Hall currents hasten the onset of convection. The medium permeability and couple-stress both postpone and hasten the onset of convection depending on the Hall parameter M. The stable solute gradient and the magnetic field (and corresponding Hall currents) are found to introduce oscillatory modes in the system, which were non-existent in their absence. The sufficient conditions for the non-existence of overstability are also obtained

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1. INTRODUCTION

The theoretical and experimental results of the onset of thermal instability (Bénard convection) in a fluid layer under varying assumptions of hydrodynamics and hydromagnetics has been treated in detail by [1] in his celebrated monograph. The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been considered [2]. Physics is quite similar in the stellar case in that helium acts like salt in raising the density and in diffusing more slowly than heat. The conditions under which convective motions are important in stellar atmospheres are usually far removed from consideration of a single component fluid and rigid boundaries, and therefore it is desirable to consider a fluid acted on by a solute gradient and free boundaries. The problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its applications to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layer of the atmosphere. The thermosolutal convection problems also arise in oceanography, limnology and engineering. [3] did the pioneering work regarding the investigation of thermosolutal convection. This work was elaborated in different physical situations [4, 5]. A double–diffusive instability that occurs when a solution of a slowly diffusing protein is layered over a denser solution of more rapidly diffusing sucrose This instability, which is deleterious to certain biochemical separations, can be suppressed by rotation in the ultra centrifuge [7].

The study of a layer of fluid heated from below in porous media is motivated both theoretically and by its practical applications in engineering. Among the applications in engineering disciplines one can find the food process industry, chemical process industry, solidification and centrifugal casting of metals. The development of geothermal power resources has increased general interest in the properties of convection in porous medium. The formation and derivation of the basic equations of a layer of a fluid heated from below in porous medium, using Boussinesq approximation, has been given in [8]. When a fluid permeates an isotropic and homogeneous porous medium, the gross effect is represented by the Darcy's law. An extensive and updated account of convection in porous media has been given [9]. The forced convection in fluid saturated porous medium channel has been studied by [10]. The free convection heat transfer of alumina-water nanofluid in a square cavity is simulated, employing the finite volume technique [11]. The flow of a non-Newtonian power-law fluid with viscous dissipation through a pipe with a variable expand ration studied in [12] and investigated the influence of the power law index, expand ratio, Darcy number and Brinkman number on heat transfer and thermodynamics irreversibility.

The effect of a magnetic field on the stability of such a flow is of interest in geophysics, particularly in the study of the earth's core, where the earth's mantle, which consists of conducting fluid, behaves like a porous medium that can become convectively unstable as a result of differential diffusion. Another application of the results of flow through a porous medium in the presence of a magnetic field is in the study of the stability of convective geothermal flow. MHD finds vital applications n MHD generators, MHD flow-meters and pumps for pumping liquid metals in metallurgy, geophysics, MHD couplers and bearings and physiological processes such as magnetic therapy.

The theory of couple-stress fluid has been formulated by [13]. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrications of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee and hip joints are the loaded–bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear. Normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish. According to the theory of [13], couple-stresses appear in noticeable magnitudes in fluids with very large molecules.

Many of the flow problems in fluids with couple-stresses, discussed by Stokes, indicate some possible experiments, which could be used for determining the material constants, and the results are found to differ from those of Newtonian fluid. Couple-stresses are found to appear in noticeable magnitudes in polymer solutions for force and couple-stresses. This theory is developed in an effort to examine the simplest generalization of the classical theory, which would allow polar effects. The constitutive equations proposed by [13] are:

$$T_{(ij)} = (-p + \lambda D_{kk})\delta_{ij} + 2\mu D_{ij} ,$$

$$T_{[ij]} = -2\eta \vec{W}_{ij.kk} - \frac{\rho}{2}\vec{\varepsilon}_{ijs} G_s ,$$

and

$$M_{ij} = 4\eta\vec{\omega}_{j,i} + 4\eta'\vec{\omega}_{i,j} ,$$

where

$$D_{ij} = \frac{1}{2}\left(V_{i,j} + V_{j,i}\right), \ \vec{W}_{ij} = -\frac{1}{2}\left(V_{i,j} - V_{j,i}\right)$$

and $\vec{\omega}_i = \frac{1}{2}\vec{\varepsilon}_{ijk} V_{k,j}$.

Here T_{ij}, $T_{(ij)}$, $T_{[ij]}$, M_{ij}, D_{ij} , $\vec{W}_{i,j}$, $\vec{\omega}_i$, G_s, $\vec{\varepsilon}_{ijk}$, V, ρ and λ, μ, η, η', are stress tensor, symmetric part of T_{ij}, anti-symmetric part of T_{ij}, the couple-stress tensor, deformation tensor, the vorticity tensor, the vorticity vector, body couple, the alternating unit tensor, velocity field, the density and material constants respectively. The dimensions of λ and μ are those of viscosity whereas the dimensions of η and η' are those of momentum.

Since the long chain hyaluronic acid molecules are found as additives in synovial fluids, [14] modeled the synovial fluid as a couple-stress fluid. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical implications. Practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The efficiency of the physiological joint lubrication is caused by several mechanisms. The synovial fluid is, due to its content of the hyaluronic acid, a fluid of high viscosity, near to a gel. The hydromagnetic stability of an unbounded couple-stress binary fluid mixture under rotation with vertical temperature and concentration gradients has been studied [15]. A couple-stress fluid with suspended particles heated from below has been investigated [16] and have found that for stationary convection, couple-stress has a stabilizing effect whereas suspended particles have a destabilizing effect. In another study, [17] have considered a couple stress fluid heated from below in a porous medium in the presence of a magnetic field and rotation. The thermal instability of a layer of a couple–stress fluid acted on by a uniform rotation has been considered [18], and have found that for stationary convection, the rotation has a stabilizing effect whereas couple-stress has both stabilizing and destabilizing effects.

The problems of couple-stress fluid heated from below in porous medium in presence of magnetic field and rotation, separately, have been studied [19, 20]. The Hall effect is likely to be important in many geophysical situations as well as in flow of laboratory plasma. There is growing importance of non-Newtonian fluids in chemical technology, industry and geophysical fluid dynamics. The Hall currents have relevance and importance in geophysics, MHD generator and industry.

Keeping in mind the importance of non-Newtonian fluids, convection in fluid layer heated and soluted from below, porous medium, magnetic field and Hall currents, the present paper attempts to study the couple-stress fluid heated and soluted from below in porous medium in the presence of uniform horizontal magnetic field to include the effect of Hall currents. The study is motivated by a model of synovial fluid. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical implications-practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The extremal efficiency of the physiological joint lubrication is caused by more mechanisms. The synovial fluid is due to the content of the hyaluronic acid a fluid of high viscosity, near to a gel. A layer of such fluid heated and soluted from below in porous medium under the action of magnetic field may find applications in physiological processes e.g. MHD finds applications in physiological processes such as magnetic therapy; heating may find applications in physio-therapy.

2. STRUCTURE OF THE PROBLEM AND BASIC EQUATIONS

Consider an infinite, horizontal, incompressible, electrically conducting couple-stress fluid layer of thickness d, heated from below so that, the temperatures and densities at the bottom surface $z=0$ are T_0 and ρ_0 and at the upper surface $z=d$ are T_d and ρ_d respectively, and that a uniform temperature gradient $\beta(=|dT/dz|)$ and a uniform solute gradient $\beta^{'}(|dC/dz|)$ are maintained. The gravity field $\vec{g}(0,0,-g)$and a uniform horizontal magnetic field $\vec{H}(H,0,0)$ pervade the system. This fluid layer is flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1.
Let $p,\rho,T,C,\alpha,\alpha^{'},g,\eta,\mu_e,N,e\ and\ \vec{q}(u,v,w)$ denote, respectively, the fluid pressure, density, temperature, solute concentration, thermal coefficient of expansion, an analogous solvent coefficient of expansion, gravitational acceleration, resistivity, magnetic permeability, electron number density, charge of an electron and fluid velocity. The equations expressing the conservation of momentum, mass, temperature, solute concentration and equation of state of couple-stress fluid through porous medium [8, 13] are

$$\frac{1}{\varepsilon}\left[\frac{\partial\vec{q}}{\partial t}+\frac{1}{\varepsilon}(\vec{q}.\nabla)\vec{q}\right]=-\frac{1}{\rho_0}\nabla p+\vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right)-\frac{1}{k_1}\left(\nu-\frac{\mu^{'}}{\rho_0}\nabla^2\right)\vec{q}+\frac{\mu_e}{4\pi\rho_0}(\nabla\times\vec{H})\times\vec{H},\quad (1)$$

$$\nabla.\vec{q}=0,\quad (2)$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=\kappa\nabla^2 T,\quad (3)$$

$$E^{'}\frac{\partial C}{\partial t}+(\vec{q}.\nabla)C,\quad (4)$$

$$\rho=\rho_0[1-\alpha(T-T_0)+\alpha^{'}(C-C_0)],\quad (5)$$

where the suffix zero refers to values at the reference level $z=0$ and in writing equation (1), use has been made of the Boussinesq approximation which states that the density variations are ignored in all terms in the equation of motion except the external force term. The magnetic permeability μ_e, the kinematic viscosity ν, the kinematic viscoelasticity $\nu^{'}$, the thermal diffusivity κ and the solute diffusivity $\kappa^{'}$ are all assumed to be constants.
The Maxwell's equations yield

$$\varepsilon\frac{d\vec{H}}{dt}=(\vec{H}.\nabla)\vec{q}+\varepsilon\eta\nabla^2\vec{H}-\frac{c\varepsilon}{4\pi Ne}\nabla\times\left[(\nabla\times\vec{H})\times\vec{H}\right],\quad (6)$$

$$\nabla.\vec{H}=0,\quad (7)$$

where $\frac{d}{dt}=\frac{\partial}{\partial t}+\varepsilon^{-1}\vec{q}.\nabla$ stand for the convection derivative.
Here $E=\varepsilon+(1-\varepsilon)\left(\frac{\rho_s c_s}{\rho_0 c_i}\right)$ is a constant and $E^{'}$ is a constant analogous to E but corresponding to solute rather than heat.ρ_s,c_s and ρ_0,c_i stand for density and heat capacity of solid (porous matrix) material and fluid, respectively. The steady state solution is

$$\vec{q}=(0,0,0),T=-\beta z,C=-\beta^{'}z+C_0,\rho=\rho_0(1+\alpha\beta z-\alpha^{'}\beta^{'}z).\quad (8)$$

Here we use linearized stability theory and normal mode analysis method. Assume small perturbations around the basic solution, and let $\delta\rho, \delta p, \theta, \gamma, \vec{q}(u,v,w)$ and $\vec{h}(h_x,h_y,h_z)$ denote respectively the perturbations in fluid density ρ, pressure p, temperature T, solute concentration C, velocity $(0,0,0)$ and magnetic field $\vec{H}(H,0,0)$. The change in density $\delta\rho$, caused by the perturbations θ and γ in temperature and concentration, is given by

$$\delta\rho = -\rho_0(\alpha\theta - \alpha'\gamma). \tag{9}$$

Then the linearized perturbation equations of the couple-stress fluid become

$$\frac{1}{\varepsilon}\frac{\partial\vec{q}}{\partial t} = -\frac{1}{\rho_0}(\nabla\delta p) - \vec{g}(\alpha\theta - \alpha'\gamma) - \frac{1}{k_1}\left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\vec{q} + \frac{\mu_e}{4\pi\rho_0}(\nabla\times\vec{h})\times\vec{H}, \tag{10}$$

$$\nabla.\vec{q} = 0, \tag{11}$$

$$E\frac{\partial\theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \tag{12}$$

$$E'\frac{\partial\gamma}{\partial t} = \beta' w + \kappa'\nabla^2\gamma, \tag{13}$$

$$\varepsilon\frac{\partial\vec{h}}{\partial t} = (\vec{H}.\nabla)\vec{q} + \varepsilon\eta\nabla^2\vec{h} - \frac{c\varepsilon}{4\pi Ne}\nabla\times[(\nabla\times\vec{h})\times\vec{H}], \tag{14}$$

$$\nabla.\vec{h} = 0. \tag{15}$$

3. THE DISPERSION RELATION

For obtaining the dispersion relation, we now analyzing the disturbances into normal modes, assuming that the perturbation quantities are of the form

$$[w, h_z, \theta, \gamma, \zeta, \xi] = [W(z), K(z), \Theta(z), \Gamma(z), Z(z), X(z)]exp(ik_x x + ik_y y + nt), \tag{16}$$

where k_x, k_y are the wave numbers along the $x-$ and $y-$ directions respectively, $k = \sqrt{(k_x^2 + k_y^2)}$ is the resultant wave number and n is the growth rate, which is, in general, a complex constant. Here $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and $\xi = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}$ stand for the z-components of vorticity and current density, respectively.

Expressing the coordinates x, y, z in the new unit of length d and letting $a = kd, \sigma = \frac{nd^2}{\nu}, p_1 = \frac{\nu}{\kappa}, p_2 = \frac{\nu}{\eta}, q = \frac{\nu}{\kappa'}, P_l = \frac{k_1}{d^2}, F = \frac{\mu'/(\rho_0 d^2)}{\nu}$ and $D = \frac{d}{dz}$; equations (10)-(15), using (16), yield

$$(D^2 - a^2)\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_l} - \frac{F}{P_l}(D^2 - a^2)\right]W - \frac{ik_x\mu_e H d^2}{4\pi\rho_0\nu}(D^2 - a^2)K + \frac{ga^2d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) = 0, \tag{17}$$

$$\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_l} - \frac{F}{P_l}(D^2 - a^2)\right]Z = \frac{ik_x\mu_e H d^2}{4\pi\rho_0\nu}X, \tag{18}$$

$$(D^2 - a^2 - p_2\sigma)K = -\left(\frac{ik_x H d^2}{\eta\varepsilon}\right)W + \frac{ick_x H d^2}{4\pi Ne\eta}X, \tag{19}$$

$$(D^2 - a^2 - p_2\sigma)X = -\left(\frac{ik_x H d^2}{\eta\varepsilon}\right)Z - \frac{ick_x H}{4\pi Ne\eta}(D^2 - a^2)K, \tag{20}$$

$$(D^2 - a^2 - Ep_1\sigma)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W, \tag{21}$$

$$(D^2 - a^2 - E'q\sigma)\Gamma = -\left(\frac{\beta' d^2}{\kappa'}\right)W. \tag{22}$$

Consider the case where both boundaries are free as well as perfect conductors of both heat and solute concentration, while the adjoining medium is perfectly conducting. The case of two free boundaries is a little artificial but it enables us to find analytical solutions and to make some qualitative conclusions. The appropriate boundary conditions, with respect to which equations (17)-(22) must be solved, are

$$W = D^2W = X = DZ = 0, \Theta = 0, \Gamma = 0, at\ z = 0\ and\ 1$$

$DX = 0, K = 0$ on a perfectly conducting boundary

and $X = 0, h_x, h_y, h_z$ are continuous with an external vacuum field on a non-conducting boundary.

(23)

The case of two free boundaries, though little artificial, is the most appropriate for stellar atmospheres [21]. Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for $z = 0$ and 1 and hence the proper solution W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{24}$$

where W_0 is a constant.

Eliminating Θ, Γ, K, Z and X between equations (17)-(22) and substituting the proper solution (24) in the resultant equation, we obtain the dispersion relation

$$R_1 = \left(\frac{1+x}{x}\right)\left[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{\pi^2 F(1+x)}{P}\right][1 + x + iEp_1\sigma_1] + S_1\frac{(1 + x + iEp_1\sigma_1)}{(1 + x + iE'q\sigma_1)}$$
$$+ Q_1 cos^2\theta \frac{(1+x)[1 + x + iEp_1\sigma_1]\left\{\left(\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{\pi^2F(1+x)}{P}\right)\begin{bmatrix}1 + x + \\ ip_2\sigma_1\end{bmatrix} + Q_1 x cos^2\theta\right\}}{\left\{\begin{gathered}\left[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{\pi^2F(1+x)}{P}\right][1 + x + ip_2\sigma_1]^2 \\ + Q_1 x cos^2\theta[1 + x + ip_2\sigma_1] \\ + M x cos^2\theta(1+x)\left[\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{\pi^2F(1+x)}{P}\right]\end{gathered}\right\}} \tag{25}$$

where

$$R_1 = \frac{g\alpha\beta d^4}{\nu\kappa\pi^4}, S_1 = \frac{g\alpha'\beta' d^4}{\nu\kappa'\pi^4}, Q_1 = \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta\varepsilon\pi^2}\ , M = \left(\frac{cH}{4\pi Ne\eta}\right)^2\ , x = \frac{a^2}{\pi^2}, i\sigma_1 = \frac{\sigma}{\pi^2}\ ,$$
$$k_x = kcos\theta and P = \pi^2 P_l\ .$$

Equation (25) is the required dispersion relation including the effects of magnetic field, Hall currents, stable solute gradient and medium permeability on a layer of couple-stress fluid heated and soluted from below in porous medium in the presence of a uniform horizontal magnetic field and Hall currents.

4. IMPORTANT THEOREMS AND DISCUSSION

THEOREM 1: For stationary convection case

(I) The magnetic field and stable solute gradient postpone the onset of convection.

(II) Hall currents hasten the onset of convection.

(III) The medium permeability hastens the onset of convection whereas the couple-stress parameter postpones the onset of convection.

Proof: When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (25) reduces to

$$R_1 = \left(\frac{1+x}{x}\right)\frac{\left(\frac{(1+x)\{1+\pi^2F(1+x)\}}{P} + Q_1xcos^2\theta\right)^2 + \frac{Mxcos^2\theta(1+x)\{1+\pi^2F(1+x)\}^2}{P^2}}{\left\{\frac{(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P} + Q_1xcos^2\theta\right\}} + S_1\ , \qquad (26)$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters S_1, Q_1, M, F and P.

To study the effects of stable solute gradient, horizontal magnetic field, Hall currents, medium permeability and couple-stress parameter, we examine the natures of $\frac{dR_1}{dS_1}, \frac{dR_1}{dQ_1}, \frac{dR_1}{dM}, \frac{dR_1}{dP}$ and $\frac{dR_1}{dF}$, analytically.

(I) Equation (26) yields

$$\frac{dR_1}{dS_1} = +1, \qquad (27)$$

$$\frac{dR_1}{dQ_1} = \left(\frac{1+x}{x}\right)xcos^2\theta\frac{\left[\frac{(1+x)\{1+\pi^2F(1+x)\}}{P} + Q_1xcos^2\theta + \frac{Mxcos^2\theta\{1+\pi^2F(1+x)\}Q_1xcos^2\theta}{P\left(\frac{(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P} + Q_1xcos^2\theta\right)}\right]}{\left(\frac{(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P} + Q_1xcos^2\theta\right)},$$

which are positive. Thus, the stable solute gradient and magnetic field postpone the onset of convection.

(II) Equation (26) also yields

$$\frac{dR_1}{dM} = -Q_1xcos^4\theta(1+x)\frac{\left(\frac{(1+x)\{1+\pi^2F(1+x)\}}{P} + Q_1xcos^2\theta\right)\left(\frac{\{1+\pi^2F(1+x)\}}{P}\right)}{\left(\frac{(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P} + Q_1xcos^2\theta\right)^2}, \qquad (29)$$

which is negative. Hence, for stationary convection, the Hall currents hasten the onset of convection on the thermosolutal instability of couple-stress fluid in porous medium in hydromagnetics in the presence of Hall currents.

(III) It is evident from equation (26) that

$$\frac{dR_1}{dP} = -\frac{(1+x)\{1+\pi^2 F(1+x)\}}{xP^2} \frac{\left[\begin{array}{c}\left(\frac{1+x}{P^2}\right)(1+x+Mxcos^2\theta)^2\{1+\pi^2F(1+x)\}^2 \\ +\frac{2Q_1xcos^2\theta(1+x)(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P} \\ +Q_1^2x^2cos^4\theta(1+x-Mxcos^2\theta)\end{array}\right]}{\left[\frac{(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P}+Q_1xcos^2\theta\right]^2}, \quad (30)$$

$$\frac{dR_1}{dF} = \left(\frac{(1+x)^2\pi^2}{xP}\right) \frac{\left[\begin{array}{c}\left(\frac{1+x}{P^2}\right)(1+x+Mxcos^2\theta)^2\{1+\pi^2F(1+x)\}^2 \\ +\frac{2Q_1xcos^2\theta(1+x)(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P} \\ +Q_1^2x^2cos^4\theta(1+x-Mxcos^2\theta)\end{array}\right]}{\left[\frac{(1+x+Mxcos^2\theta)\{1+\pi^2F(1+x)\}}{P}+Q_1xcos^2\theta\right]^2}, \quad (31)$$

Hence, it is clear from (30) and (31) that, for stationary convection, the medium permeability hastens the onset of convection whereas, the couple-stress postpones the onset of convection on the thermal instability of couple-stress fluid in porous medium in hydromagnetics in the presence of Hall currents for all wave numbers

$$(1+x) > Mxcos^2\theta$$

which is normally satisfied as the Hall currents parameter M is very small compared to unity.

Theorem 1 is also proved numerically as follow:

In Figure 1, R_1 is plotted against x for $S_1 = 10, 20, 30; P = 50, \theta = 45^0, F = 2, Q_1 = 10$ and $M = 10$. It is clear that the stable solute gradient postpones the onset of convection in a couple-stress fluid heated and soluted from below in a porous medium in the presence of Hall currents as the Rayleigh number increases with the increase in stable solute gradient parameter.

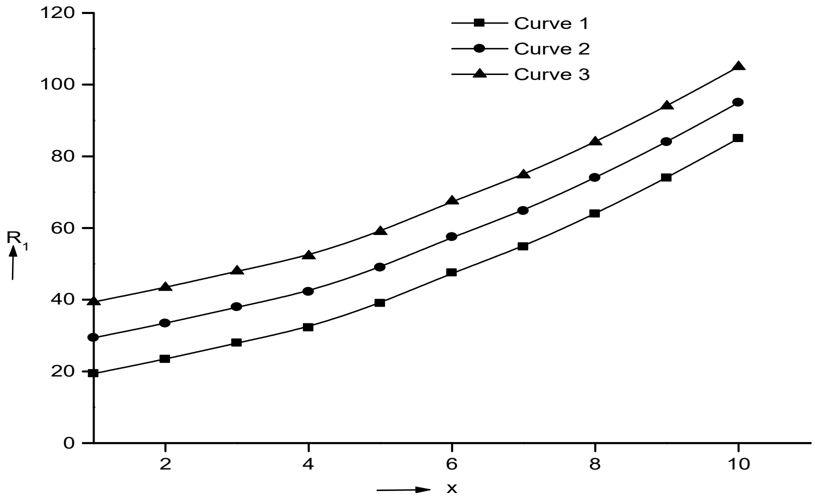

Figure 1. The variation of Rayleigh number (R_1) with wave number (x) for $P = 50, F = 2, \theta = 45^0, M = 10, Q_1 = 10$; $S_1 = 10$ for curve 1, $S_1 = 20$ for curve 2 and $S_1 = 30$ for curve 3.

In Figure 2, R_1 is plotted against x for $Q_1 = 10, 20, 30; P = 50, \theta = 45^0, F = 2, S_1 = 10$ and $M = 10$. It is clear that the magnetic field postpones the onset of convection in a couple-stress fluid heated from below in a porous medium in the presence of Hall currents as the Rayleigh number increases with the increase in magnetic field parameter.

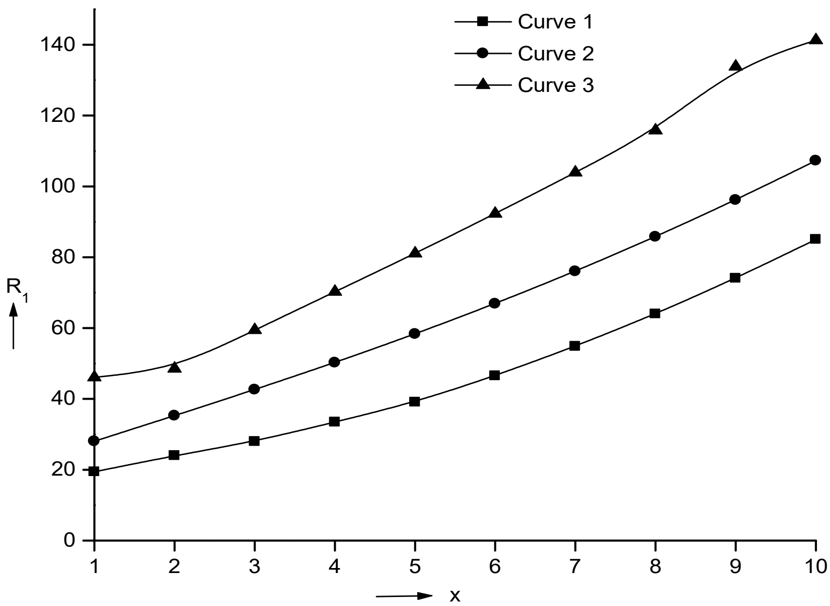

Figure 2. The variation of Rayleigh number (R_1) with wave number (x) for $P = 50, F = 2, \theta = 45^0, M = 10, S_1 = 10$; $Q_1 = 10$ for curve 1, $Q_1 = 20$ for curve 2 and $Q_1 = 30$ for curve 3.

In Figure 3, R_1 is plotted against x for $M = 10, 20, 30; P = 50, \theta = 45^0, F = 2, S_1 = 10$ and $Q_1 = 10$. Here we find that the Hall currents hastens the onset of convection for all wave numbers as the Rayleigh number decreases with the increase in the Hall currents parameter.

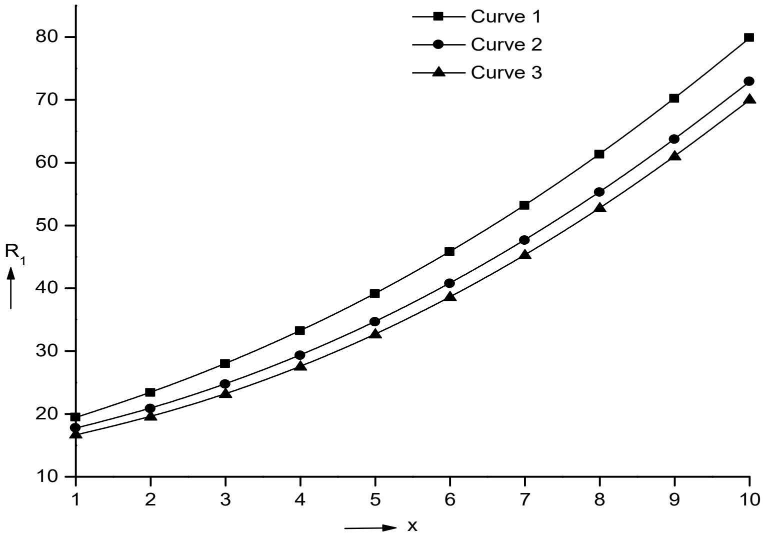

Figure 3. The variation of Rayleigh number (R_1) with wave number (x) for $P = 50, F = 2, \theta = 45^0, Q_1 = 10, S_1 = 10$; M= 10 for curve 1, M= 20 for curve 2 and M= 30 for curve 3.

In Figure 4, R_1 is plotted against x for R_1 is plotted against x for $P = 10, 20, 30; M = 0.1, \theta = 45^0, F = 2, S_1 = 10$ and $Q_1 = 100$. It is clear that when $M \ll 1$, the medium permeability always hastens the onset of convection for all wave numbers as the Rayleigh number decreases with an increase in medium permeability parameter.

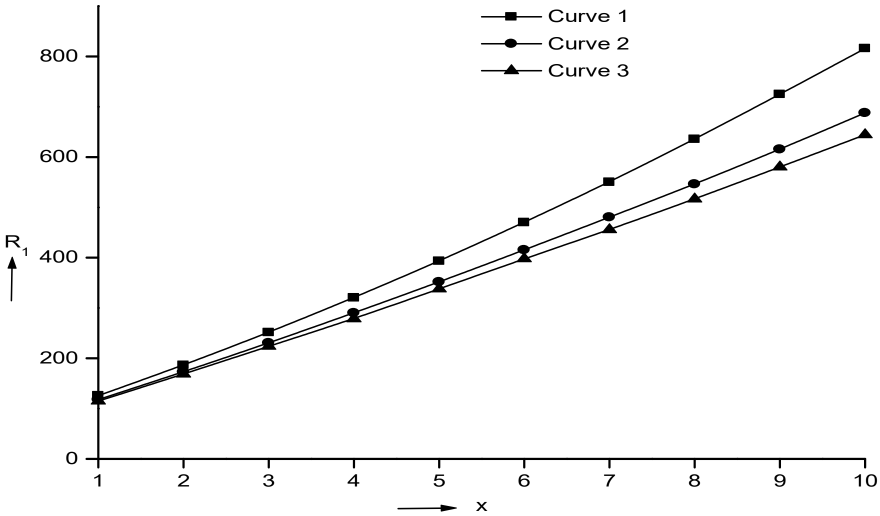

Figure 4. The variation of Rayleigh number (R_1) with wave number (x) for $F = 2, Q_1 = 100, \theta = 45^0, M = 0.1, S_1 = 10$; P= 10 for curve 1, P= 20 for curve 2 and P= 30 for curve 3.

In Figure 5, R_1 is plotted against x for $P = 10, 20, 30; M = 100, \theta = 45^0, F = 2, S_1 = 10$ and $Q_1 = 100$. Here we find that when $M > 1$, the medium permeability postpones the onset of convection for small wave numbers only as the Rayleigh number increases with an increase in medium permeability parameter and hastens the onset of convection for higher wave numbers as the Rayleigh number decreases with an increase in medium permeability parameter.

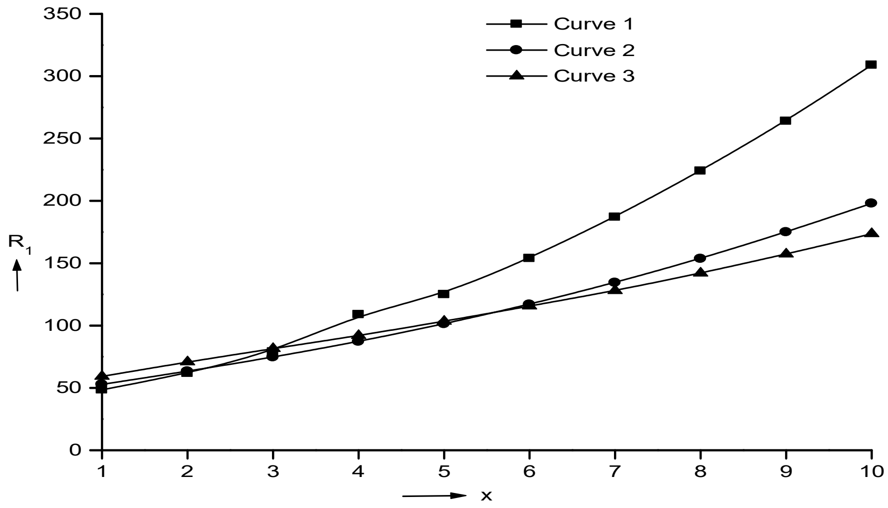

Figure 5. The variation of Rayleigh number (R_1) with wave number (x) for $F = 2, Q_1 = 100, \theta = 45^0, M = 100, S_1 = 10$; P= 10 for curve 1, P= 20 for curve 2 and P= 30 for curve 3.

In Figure 6, R_1 is plotted against x for $F = 1,2,3,4$; $Q_1 = 100, \theta = 45^0, P = 10, S_1 = 10$ and $M = 0.1$. It is clear that when $M \ll 1$, the couple-stress postpones the onset of convection for all wave numbers as the Rayleigh number increases with the increase in couple-stress parameter.

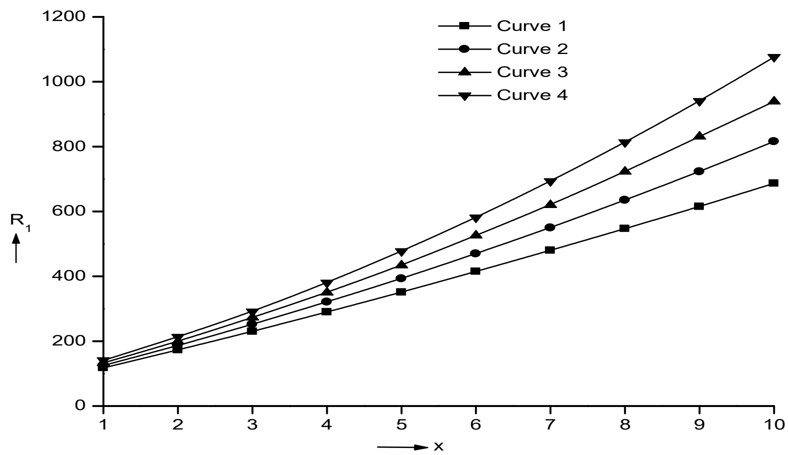

Figure 6. The variation of Rayleigh number (R_1) with wave number (x) for P= $10, \theta = 45^0, M = 0.1, Q_1 = 100, S_1 = 10$; F= 1 for curve 1, F= 2 for curve 2, F= 3 for curve 3 and $F = 4$ for curve 4..

In Figure 7, R_1 is plotted against x for $F = 1,2,3$; $Q_1 = 100, \theta = 45^0, P = 10, S_1 = 10$ and $M = 100$. It is clear that when $M > 1$, the couple-stress hastens the onset of convection for small wave numbers as the Rayleigh number decreases with the increase in couple-stress parameter and postpones the onset of convection for higher wave numbers as the Rayleigh number increases with the increase in couple-stress parameter.

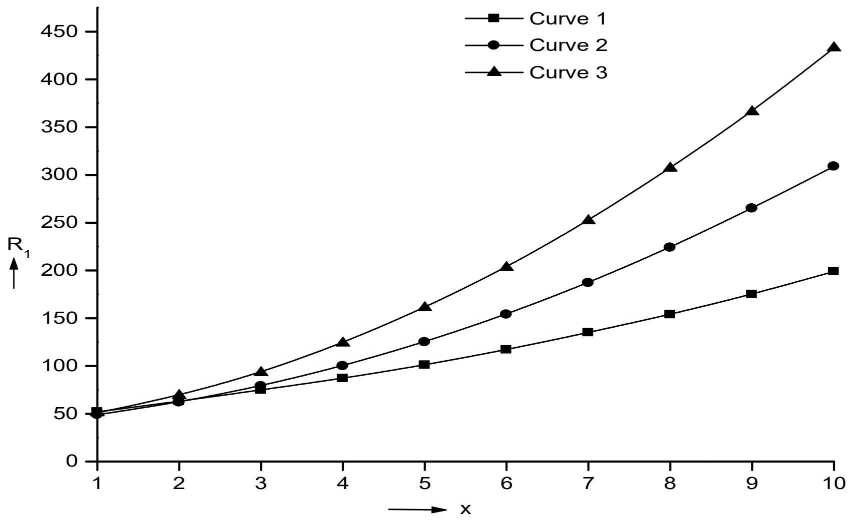

Figure 7. The variation of Rayleigh number (R_1) with wave number (x) for P= $10, \theta = 45^0, M = 100, Q_1 = 100, S_1 = 10$; F= 1 for curve 1, F= 2 for curve 2 and F= 3 for curve 3.

THEOREM 2: The system is stable or unstable.

Proof: Multiplying equation (17) by W^*, which is the complex conjugate of W, and using equations (18)-(22) together with the boundary conditions (23), we obtain

$$FI_1 + \left(1 + P_l \frac{\sigma}{\varepsilon}\right) I_2 - \left(\frac{g\alpha\kappa a^2}{\nu\beta} P_l\right) [I_3 + Ep_1\sigma^* I_4] + \left(\frac{g\alpha'\kappa' a^2}{\nu\beta'}\right) [I_5 + E' q\sigma^* I_6]$$
$$+ \frac{\mu_e \varepsilon\eta}{4\pi\rho_0\nu} P_l [I_7 + p_2\sigma^* I_8] + \frac{\mu_e \varepsilon\eta d^2}{4\pi\rho_0\nu} P_l [I_{11} + p_2\sigma I_{12}]$$
$$+ d^2 \left[\left(1 + P_l \frac{\sigma^*}{\varepsilon}\right) I_{10} + FI_9\right] = 0\,, \tag{32}$$

where

$$I_1 = \int_0^1 (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)dz\,,$$

$$I_2 = \int_0^1 (|DW|^2 + a^2|W|^2)dz\,,$$

$$I_3 = \int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz,$$

$$I_4 = \int_0^1 (|\Theta|^2)dz\,, \quad I_5 = \int_0^1 (|D\Gamma|^2 + a^2|\Gamma|^2)dz\,,$$

$$I_6 = \int_0^1 (|\Gamma|^2)dz\,,$$

$$I_7 = \int_0^1 (|D^2K|^2 + 2a^2|DK|^2 + a^4|K|^2)dz,$$

$$I_8 = \int_0^1 (|DK|^2 + a^2|K|^2)dz,$$

$$I_9 = \int_0^1 (|DZ|^2 + a^2|Z|^2)dz\,,$$

$$I_{10} = \int_0^1 (|Z|^2) dz\,,$$

$$I_{11} = \int_0^1 (|DX|^2 + a^2|X|^2) dz\,,$$

$$I_{12} = \int_0^1 (|X|^2) dz\,. \tag{33}$$

The integrals $I_1\,, \dots \dots, I_{12}$ are all positive definite. Putting $= \sigma_r + i\sigma_i$, where $\sigma_r\,, \sigma_i$ are real and equating the real and imaginary parts of equation (32), we obtain

$$\sigma_r \left[\frac{I_2}{\varepsilon} - \frac{g\alpha\kappa a^2}{\nu\beta} Ep_1 I_4 + \frac{g\alpha'\kappa' a^2}{\nu\beta'} E' q I_6 + \frac{\mu_e \varepsilon\eta}{4\pi\rho_0\nu} p_2 (I_8 + d^2 I_{12}) + \frac{d^2}{\varepsilon} I_{10}\right] = -\left[\frac{F}{P_l} I_1 + \frac{1}{P_l} I_2 - \frac{g\alpha\kappa a^2}{\nu\beta} I_3 + \frac{g\alpha'\kappa' a^2}{\nu\beta'} I_5 + \frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}(I_7 + d^2 I_{11}) + \frac{d^2}{P_l}(I_{10} + F I_9)\right], \tag{34}$$

$$\sigma_i \left[\frac{I_2}{\varepsilon} + \frac{g\alpha\kappa a^2}{\nu\beta} Ep_1 I_4 - \frac{g\alpha'\kappa' a^2}{\nu\beta'} E' q I_6 - \frac{\mu_e \varepsilon\eta}{4\pi\rho_0\nu} p_2 (I_8 - d^2 I_{12}) - \frac{d^2}{\varepsilon} I_{10}\right] = 0\,. \tag{35}$$

It is evident from equation (34) that σ_r is either positive or negative. The system is, therefore, either stable or unstable.

THEOREM 3: The modes may be oscillatory or non-oscillatory in contrast to case of no magnetic field, and in the absence of stable solute gradient where modes are non-oscillatory.

Proof: Equation (35) yields that σ_i may be either zero or non-zero, meaning that the modes may be either non-oscillatory or oscillatory. In the absence of stable solute gradient and magnetic field, equation (35) reduces to

$$\left[\frac{I_2}{\varepsilon} + \frac{g\alpha\kappa a^2}{\nu\beta} Ep_1 I_4\right]\sigma_i = 0\,, \tag{36}$$

and the terms in brackets are positive definite. Thus $\sigma_i = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for a porous medium, in the absence of stable solute gradient and magnetic field. This result is true for the porous as well as non-porous medium as studied in [1]. The oscillatory modes are introduced due to the presence of the stable solute gradient and the magnetic field (and corresponding Hall currents), which were non-existent in their absence.

THEOREM 4: The system is stable for $\frac{g\alpha\kappa}{\nu\beta}\frac{P_l}{F} \le \frac{27}{4}^{4}$ and under the condition $\frac{g\alpha\kappa}{\nu\beta}\frac{P_l}{F} > \frac{27}{4}^{4}$, the system becomes unstable.

Proof: From equation (35), it is clear that σ_i is zero when the quantity multiplying it is not zero and arbitrary when this quantity is zero.

If $\sigma_i \neq 0$, equation (34) upon utilizing (35) and the Rayleigh-Ritz inequality gives

$$\left[\frac{27\pi^4}{4}-\frac{g\alpha\kappa}{\nu\beta}\frac{P_l}{F}\right]\int_0^1|W|^2\,dz+\frac{(\pi^2+a^2)}{a^2}\frac{P_l}{F}\left\{\begin{matrix}\frac{\mu_e\varepsilon\eta}{2\pi\rho_0\nu}p_2d^2\sigma_rI_{12}+\frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}d^2I_{11}\\+\frac{d^2}{P_l}I_{10}+\frac{d^2F}{P_l}I_9+\\\frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}I_7+\frac{g\alpha'\kappa'a^2}{\nu\beta'}I_5+\\\frac{1}{P_l}I_2+\frac{2\sigma_r}{\varepsilon}I_2\end{matrix}\right\}\le 0, \tag{37}$$

since minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ with respect to a^2 is $\frac{27\ \ 4}{4}$.
Now, let $\sigma_r\ge 0$, we necessarily have from inequality (37) that

$$\frac{g\alpha\kappa}{\nu\beta}\frac{P_l}{F}>\frac{27\pi^4}{4}. \tag{38}$$

Hence, if

$$\frac{g\alpha\kappa}{\nu\beta}\frac{P_l}{F}\le\frac{27\pi^4}{4}, \tag{39}$$

Then $\sigma_r<0$. Therefore, the system is stable.
Thus, under condition (39), the system is stable and under condition (38) the system becomes unstable

THEOREM 5: The sufficient conditions for the non-existence of overstability are

$$\kappa<min\left(E\eta,\frac{E\kappa'}{E'}\right)\ \ and\ \ \nu>max\left[\frac{\rho_0k_x^2}{\mu'}\left(\frac{cHk_1}{4\varepsilon Ne}\right)^2,\left(\frac{\mu_e}{2\pi\rho_0}\right)^{1/2}\left(\frac{k_1Hk_x}{\varepsilon}\right)\right].$$

Proof: Here we discuss the possibility of whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (25) will admit of solutions with σ_1 real.
Equating real and imaginary parts of equation (25) and eliminating R_1 between them, we obtain

$$A_4c_1^4+A_3c_1^3+A_2c_1^2+A_1c_1+A_0=0\,, \tag{40}$$

where we have put $c_1=\sigma_1^2$, $b=1+x$ and

$$A_4=\frac{E'^2q^2p_2^4}{\varepsilon^2}\left[\left(\frac{1}{\varepsilon}+\frac{Ep_1\pi^2F}{P}\right)b+\frac{Ep_1}{P}\right]\,, \tag{41}$$

$$A_3 = \left[E'^2 q^2 \left(\frac{1}{\varepsilon} + \frac{Ep_1\pi^2 F}{P}\right)\left(\frac{p_2^2\pi^4F^2}{P^2} + \frac{2}{\varepsilon^2}\right)p_2^2 + \frac{p_2^4}{\varepsilon^2}\left(\frac{1}{\varepsilon} + \frac{Ep_1\pi^2F}{P}\right)\right]b^3$$
$$+ \left[2E'^2 q^2 \left(\frac{1}{\varepsilon} + \frac{Ep_1\pi^2F}{P}\right)\left(\frac{p_2^2\pi^2F}{P} - \frac{Mxcos^2\theta}{\varepsilon^2}\right)p_2^2 + \frac{p_2^4 Ep_1}{\varepsilon^2 P}\right.$$
$$\left. + \frac{E'^2 q^2 Ep_1}{P}\left(\frac{p_2^2\pi^4F^2}{P^2} + \frac{2}{\varepsilon^2}\right)p_2^2\right]b^2$$
$$+ E'^2 q^2 \left[\left(\frac{1}{\varepsilon} + \frac{Ep_1\pi^2F}{P}\right)\left(\frac{p_2}{P^2} - \frac{2Q_1xcos^2\theta}{\varepsilon}\right)p_2^3\right.$$
$$\left. + \frac{2Ep_1p_2^2}{P}\left(\frac{p_2^2\pi^2F}{P^2} - \frac{Mxcos^2\theta}{\varepsilon^2}\right) + \left(\frac{p_2^2}{\varepsilon^2}(Ep_1 - p_2)\right)\right]b$$
$$+ \left[\frac{E'^2 q^2 Ep_1}{P}\left(\frac{p_2}{P^2} - \frac{2Q_1xcos^2\theta}{\varepsilon}\right)p_2^3\right.$$
$$\left. + \left(\frac{p_2^4}{\varepsilon^2}S_1(Ep_1 - p_2)(b-1)\right)\right], \tag{42}$$

and the coefficients A_0, A_1 and A_2, being quite lengthy and not needed in the discussion of overstability, have not been written here.
Since σ_1 is real for overstability, the four values of $c_1 (= \sigma_1^2)$ are positive. The sum of roots of (40) is $-\frac{A_3}{A_4}$, and if this is to be negative, then $A_3 > 0$, $A_4 > 0$.
It is clear from (41) and (42) that A_3 and A_4 are always positive if

$$Ep_1 > p_2, Ep_1 > E'q, \frac{p_2^2\pi^2F}{P^2} > \frac{Mxcos^2\theta}{\varepsilon^2} \text{ and } \frac{p_2}{P^2} > \frac{2Q_1xcos^2\theta}{\varepsilon}, \tag{43}$$

which imply that

$$\kappa < E\eta, \kappa < \frac{E\kappa'}{E'}, \nu > \frac{\rho_0k_x^2}{\mu'}\left(\frac{cHk_1}{4\varepsilon Ne}\right)^2, \nu > \left(\frac{\mu_e}{2\pi\rho_0}\right)^{1/2}\left(\frac{k_1Hk_x}{\varepsilon}\right) \tag{44}$$

$$i.e\ \kappa < min\left(E\eta, \frac{E\kappa'}{E'}\right), \nu > max\left[\frac{\rho_0k_x^2}{\mu'}\left(\frac{cHk_1}{4\varepsilon Ne}\right)^2, \left(\frac{\mu_e}{2\pi\rho_0}\right)^{1/2}\left(\frac{k_1Hk_x}{\varepsilon}\right)\right]. \tag{45}$$

Thus $< min\left(E\eta, \frac{E\kappa'}{E'}\right), \nu > max\left[\frac{\rho_0k_x^2}{\mu'}\left(\frac{cH\ _1}{4\varepsilon Ne}\right)^2, \left(\frac{\mu_e}{2\pi\rho_0}\right)^{1/2}\left(\frac{k_1Hk_x}{\varepsilon}\right)\right]$, therefore, are the sufficient conditions for the non-existence of overstability, the violation of which does not necessarily imply occurrence of overstability.

5. CONCLUSIONS

A layer of couple-stress fluid heated and soluted from below in porous medium is considered in the presence of uniform horizontal magnetic field to include the effect of Hall currents. The inclusion of Hall currents gives rise to a cross flow i.e. a flow at right angles to the primary flow in a channel in the presence of a transverse magnetic field [22] whereas [23] found that Hall effect produces a cross-flow of double-swirl pattern in incompressible flow through a straight channel with arbitrary cross-section. This breakdown of the primary flow and formation of a secondary flow may be attributed to the inherent instability of the primary flow in the presence of Hall current. [22] pointed out that even if the distribution of the primary flow velocity were stable to external disturbances, the whole layer may become turbulent if the distribution of the cross-flow velocity is unstable. A similar situation occurs on the three-dimensional boundary layer along a swept-back wing. The presence of Hall current induces a vertical component of vorticity [24] and this may well be the reason for the destabilizing influence. The main conclusions from the analysis of this paper are as follows:

- For the stationary convection case, the stable solute gradient and magnetic field postpones the onset of convection whereas, the Hall currents hastens the onset of convection. The medium permeability hastens the onset of convection whereas the couple-stress postpones the onset of convection for all wave numbers

$$(1+x) > Mxcos^2\theta\ ,$$

which is normally satisfied as the Hall currents parameter M is very small compared to unity.

- It is found that the stable solute gradient and the magnetic field (and corresponding Hall currents) introduce oscillatory modes in the system, which were non-existent in their absence.
- It is observed that the system is stable for $\frac{g\alpha\kappa}{\nu\beta}\frac{P_l}{F} \leq \frac{27\ \ ^4}{4}$ and under the condition $\frac{g\alpha\kappa}{\nu\beta}\frac{P_l}{F} > \frac{27\pi^4}{4}$, the system becomes unstable.
- The case of overstability is also considered. The conditions

$$\kappa < min\left(E\eta, \frac{E\kappa'}{E'}\right), \nu > max\left[\frac{\rho_0 k_x^2}{\mu'}\left(\frac{cHk_1}{4\varepsilon Ne}\right)^2, \left(\frac{\mu_e}{2\pi\rho_0}\right)^{1/2}\left(\frac{k_1 H k_x}{\varepsilon}\right)\right]$$

are the sufficient conditions for the non-existence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

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