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Numerical solution of a viscoelastic contact problem with normal compliance and unilateral constraint

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ABSTRACT

A numerical method is presented for a mathematical model which describes the frictional contact between a viscoelastic body and an obstacle. The process is quasistatic, and the material's behaviour is described by means of a viscoelastic constitutive law with long memory. The contact is modelled with normal compliance condition restricted by unilateral constraint and associated to a version of Coulomb's law of dry friction. A solution algorithm is discussed and implemented. Finally, numerical simulation results are reported on a two-dimensional test problem. These simulations show the efficiency of the algorithm and the corresponding mechanical interpretations.

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1. INTRODUCTION

An important class of engineering applications consists of frictional contact problems. Therefore, the contact models are of great importance in mechanical engineering, focusing on modeling and simulating complex interactions between solid surfaces, as well as managing stress and friction. Mathematical, mechanical and numerical state of the art on contact mechanics can be found in [1-13] and the references therein. The model considered in [14] was based on a viscoelastic constitutive law with long memory, a contact conditions combining normal compliance, memory term, unilateral constraint and a frictional sliding version of the Coulomb's law.

This paper presents a continuation of [14] and is devoted to the numerical solution of the contact model introduced in [15]. In this work, we study, from the numerical point of view, a quasistatic frictional contact problem between a viscoelastic body with long memory and an obstacle. The contact is modeled with normal compliance and is associated with unilateral constraints and with version of Coulomb's law of dry friction. These nonstandard contact conditions could model the contact with the rigid foundation covered by a layer composed of a soft material.

The main novelty of this model lies in the chosen the material's behavior with a viscoelastic constitutive law with long memory and in the boundary conditions describing the contact surface. The considered model leads to a new and more interesting mathematical model, involving new operators and new functionals. The analysis and numerical approach of this system represent the main trait of novelty of the present paper. To this end, we consider a fully discrete scheme to approximate the problem, involving finite difference discretization in time and finite element discretization in space. We treat the frictional contact conditions by using a numerical approach based on the combination of the penalized method and the augmented Lagrangian method. Finally, we implement this scheme in a numerical code and present numerical simulation results on a model two-dimensional problem, illustrating the mechanical behavior related to the contact conditions.

The paper is organized as follows. In Section 2, we present a brief description of the mechanical model and its variational formulation, which consists of a system coupling a variational inequality for the displacement field and a nonlinear equation for the stress field. The numerical solution used for solving the discrete problem is described in Section 3. Our main interest lies in Section 4 where we present numerical simulations in the study of a two-dimensional test problem. Finally, in Section 5 we present some conclusions and perspectives.

2. THE MODEL

The physical setting is the following. A viscoelastic body occupies a regular domain $\Omega \subset \mathbb{R}^d, d = 2, 3$ with a smooth boundary $\partial\Omega = \Gamma$, divided into three disjoint measurable parts Γ_D, Γ_N and Γ_C such that $meas\Gamma_D > 0$. We use the notation $x = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\nu = (\nu_i)$ the outward unit normal at Γ. Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, i.e. $f_{,i} = \frac{\partial f}{\partial x_i}$.

We denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and the Euclidean norm on $\mathbb{R}^d$ and S^d are defined by $u.v = u_i v_i$, $||v|| = (v \cdot v)^{1/2}$ for $u,\ v \in \mathbb{R}^d$ and $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $||\tau|| = (\tau \cdot \tau)^{1/2}$ for $\sigma, \tau \in S^d$. Also, we denote by $u = (u_i) \in \mathbb{R}^d$ the displacement vector, by $\sigma = (\sigma_{ij}) \in S^d$ the stress tensor, by $\varepsilon(u) = (\varepsilon_{ij}(u)) \in S^d$ the linearized strain tensor, i.e. $\varepsilon_{ij}(u) = (u_{i,j} + u_{j,i})/2$. Also, we denote by u_ν and u_τ the normal and tangential components of u on Γ given by $u_\nu = u \cdot \nu$, $u_\tau = u - u_\nu \nu$. Finally, σ_ν and σ_τ will represent the normal and the tangential stress on Γ, that is $\sigma_\nu = (\sigma\nu).\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

The viscoelastic body is in equilibrium under the action of body forces of density f_0 and surface tractions of density f_N which act on Γ_N. The body is clamped on Γ_D; therefore, the displacement field vanishes there. In the reference configuration, the body is in contact over Γ_C with a deformable foundation. The frictional contact conditions are derived from the following five assumptions:

a) The foundation is made by a rigid body covered by a layer made by of deformable material, say asperities. Therefore, the penetration is restricted, i.e.

$$u_\nu \leq g, \tag{1}$$

where $g > 0$ represents the thickness of the deformable layer.

b) When there is separation, then the reaction of the obstacle vanishes, Therefore,

$$u_\nu < 0 \ \Rightarrow\ \sigma_\nu = 0,\ \ \sigma_\tau = 0. \tag{2}$$

c) When there is penetration, as far as the normal displacement does not reach the bound g, the contact is described with a normal compliance condition associated to the static version of Coulomb's law of dry friction. Therefore,

$$0 \leq u_\nu < g \ \Rightarrow \begin{cases} -\sigma_\nu = p_\nu(u_\nu), \\ \|\sigma_\tau\| \leq p_\tau(u_\nu), \\ -\sigma_\tau = p_\tau(u_\nu)\dfrac{u_\tau}{\|u_\tau\|} \ \ if\ \ u_\tau \neq 0. \end{cases} \tag{3}$$

Here $p_\nu(.)$ and $p_\tau(.)$ are non-negative prescribed functions that vanishes for negative argument.

d) When the normal displacement reaches the bound g, then the normal stress is larger than a given value $F_b > 0$ and, moreover, friction follows the static Tresca law with the friction bound F_b. Therefore,

$$u_\nu = g \ \Rightarrow \begin{cases} -\sigma_\nu \geq F_b, \\ \|\sigma_\tau\| \leq F_b, \\ -\sigma_\tau = F_b \dfrac{u_\tau}{\|u_\tau\|} \ \ if\ \ u_\tau \neq 0. \end{cases} \tag{4}$$

e) To accommodate the conditions (3) and (4) we assume the compatibility

$$F_b = p_\tau(g). \tag{5}$$

Then it is easy to see those assumptions $(1-5)$ can be written, equivalently, as follows:

$$\begin{cases} u_\nu \le g, \quad \sigma_\nu + p_\nu(u_\nu) \le 0, \quad (u_\nu - g)\big(\sigma_\nu + p_\nu(u_\nu)\big) = 0, \\ \|\sigma_\tau\| \le p_\tau(u_\nu), \quad -\sigma_\tau = p_\tau(u_\nu)\dfrac{u_\tau}{\|u_\tau\|} \ if \ u_\tau \ne 0. \end{cases} \tag{6}$$

Condition (6) assume that the behaviour of the foundation is an elastic-rigid one (see [16]).

The mechanical problem of frictional contact of the viscoelastic body with normal compliance and unilateral constraint may be formulated as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \longrightarrow S^d$, *such that*

$$\sigma = \mathcal{A}\varepsilon(u) + \int_0^t \mathcal{R}(t-s)\varepsilon(u(s))ds \qquad in\ \Omega \times (0,T), \tag{7}$$

$$Div\,\sigma + f_0 = 0 \quad in\ \Omega \times (0,T), \tag{8}$$

$$u = 0 \quad on\ \Gamma_D \times (0,T), \tag{9}$$

$$\sigma\nu = f_N \quad on\ \Gamma_N \times (0,T), \tag{10}$$

$$\begin{cases} u_\nu \le g, \quad \sigma_\nu + p_\nu(u_\nu) \le 0 \\ (u_\nu - g)\big(\sigma_\nu + p_\nu(u_\nu)\big) = 0 \end{cases} \qquad on\ \Gamma_C \times (0,T), \tag{11}$$

$$\begin{cases} \|\sigma_\tau\| \le p_\tau(u_\nu) \\ \sigma_\tau = -p_\tau(u_\nu)\frac{u_\tau}{\|u_\tau\|} \ if \ u_\tau \ne 0 \end{cases} \qquad on\ \Gamma_C \times (0,T). \tag{12}$$

In $(7-12)$ and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$ and the time variable $t \in [0,T]$, where $T > 0$.

Equations (7) represents the viscoelastic constitutive law with long memory in which $\mathcal{A}$ (possible nonlinear) and $\mathcal{R}$ are the elasticity operator and the relaxation tensor, respectively, see [1]. Equation (8) is the equilibrium equation and Div denotes the divergence operators, i.e., $Div\,\sigma = (\sigma_{ij,i})$; we use it here since we assume that process is quasistatic. Conditions (9) and (10) are the displacement and traction boundary conditions. Finally, conditions (11) and (12) represent the frictional contact condition with normal compliance and unilateral constraint associated to the Coulomb friction law, previously described in this section, see (6).

To present the variational formulation of Problem P we need some additional notation and preliminaries. We start by introducing the spaces $H = L^2(\Omega, \mathbb{R}^d)$, $\mathcal{K} = L^2(\Omega, S^d)$. The spaces H and $\mathcal{K}$ are Hilbert spaces equipped with the inner products $(u,v)_H = \int_\Omega u \cdot v\,dx$ and $(\sigma,\tau)_{\mathcal{K}} = \int_\Omega \sigma \cdot \tau\ dx$, respectively. The associated norms in H and $\mathcal{K}$ are denoted by $||\cdot||_H$ and $||\cdot||_{\mathcal{K}}$, respectively.

For the displacement fields, we introduce the Hilbert spaces $V = \{v \in H^1(\Omega, \mathbb{R}^d);\ v = 0\ on\ \Gamma_D\}$. On V, we consider the inner product and the corresponding norm given by

$$(u,v)_V = \big(\varepsilon(u), \varepsilon(v)\big)_{\mathcal{K}}, \quad ||v||_V = ||\varepsilon(v)||_{\mathcal{K}} \ \text{ for all } u,v \in V. \tag{13}$$

We introduce the set of admissible displacements defined by $U = \{v \in V;\ v_\nu \le g\ a.e.\ on\ \Gamma_C\}$. We consider the trace spaces $X_\nu = \{v_{\nu \backslash \Gamma_C} : v \in V\}$ and $X_\tau = \{v_{\tau \backslash \Gamma_C} : v \in V\}$, equipped with their usual norms. Denote by X_ν^* and X_τ^* the duals of the spaces X_ν and X_τ, respectively. Moreover, we denote by $<\cdot,\cdot>_{X_\nu^* \times X_\nu}$ and $<\cdot,\cdot>_{X_\tau^* \times X_\tau}$ the corresponding duality pairing mappings.

We consider the three mappings $P : V \longrightarrow V,\ j : V \times V \longrightarrow \mathbb{R}\ and\ f : [0,T] \longrightarrow V$, defined by

$$(Pu, v)_V = \int_{\Gamma_C} p_\nu(u_\nu) v_\nu \, da, \tag{14}$$

$$j(u, v) = \int_{\Gamma_C} p_\tau(u_\tau) \|v_\tau\| \, da, \tag{15}$$

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_N} f_N(t) \cdot v \, da, \tag{16}$$

for all $v \in V$.

Then, performing integration by parts, we obtain the following variational formulation of Problem P in terms of the displacement and the stress fields.

Problem P_V. *Find a displacement field* $u : [0,T] \longrightarrow U$ *and a stress field* $\sigma : [0,T] \longrightarrow \mathcal{K}$ *such for a.e.* $t \in (0,T)$

$$\sigma = \mathcal{A}\varepsilon(u) + \int_0^t \mathcal{R}(t-s)\varepsilon(u(s))ds, \tag{17}$$

$$\big(\sigma(t), \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{K}} + (Pu(t), v - u(t))_V + j(u(t), v) - j(u(t), u(t)) \geq (f(t), v - u(t))_V \ \forall\, v \in U. \tag{18}$$

A result of existence and uniqueness for the problem P_V was provided in [15]. Based on this previous variational formulation, our goal in the next section is to provide the numerical analysis of this non trivial contact problem.

3. NUMERICAL SOLUTION

To describe the numerical method for the variational problem P_V, we first introduce a fully discrete scheme to approximate the solution of problem P_V. Let $V^h \subset V$ be a linear finite element space on the domain, which is vanishing on the boundary Γ_D, where $h > 0$ denotes the spatial discretization parameter. We now consider the spaces $X_\nu^h = \{ v_{\nu \backslash \Gamma_C}^h \ : \ v^h \in V^h\}$ and $X_\tau^h = \{ \boldsymbol{v}_{\tau \backslash \Gamma_C}^h \ : \ \boldsymbol{v}^h \in V^h\}$ equipped with their usual norm. We also consider the discrete space of piecewise constants $Y_\nu^h \subset L^2(\Gamma_C)$ and $Y_\tau^h \subset L^2(\Gamma_C)^d$ related to the discretization of the normal and the tangential stress, respectively, and let $W^h = Y_\nu^h \times Y_\tau^h$.

Next, the time derivatives are discretized by using a uniform partition of $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N \leq T$. Let k be the time step size, $k = T/N$, and for a continuous function $f(t)$ let $f_n = f(t_n)$.

In addition, for a discretization of the history dependent operator in (7), we employ a modified trapezoidal rule to approximate the integral in the sense that on the last sub-interval $[t_{n-1}, t_n]$, the left-point rectangular rule is applied (see [17]). The approximation of $\int_0^{t_n} \mathcal{R}(t_n - s)\varepsilon(v(s))ds$ can be defined as follows:

$$\int_0^{t_n} \mathcal{R}(t_n - s)\varepsilon(v(s))ds \approx \frac{k}{2}\mathcal{R}(t_n - t_0)\varepsilon(v_0) + k\sum_{j=1}^{n-1}\mathcal{R}(t_n - t_j)\varepsilon(v_j) + \frac{k}{2}\mathcal{R}(t_n - t_{n-1})\varepsilon(v_{n-1}). \tag{19}$$

Finally, we note that the numerical treatment of the conditions (11) is based on the use of a penalization method for the normal compliance conditions part and an augmented Lagrangian method for the unilateral constraints part. For friction law (12), we use an augmented Lagrangian approach, see [18] and [19], respectively. To this end, we introduce the notation, where $\lambda_\nu = \lambda \cdot \nu$ and $\lambda_\tau = \lambda - \lambda_\nu \nu$.

Our discretized penalty and augmented Lagrangian based methods for unilateral contact problems in viscoelastics then reads:

Problem P_V^{hk}. *Find a discrete displacement* $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset V^h$, *a discrete normal stress* $\lambda_\nu^{hk} = \{\lambda_{\nu_n}^{hk}\}_{n=0}^N \subset Y_\nu^h$, *and a discrete tangential stress* $\lambda_\tau^{hk} = \{\lambda_{\tau_n}^{hk}\}_{n=0}^N \subset Y_\tau^h$ *such that for all* $n = 1, \cdots, N$

$$\left(\mathcal{A}\varepsilon(u_n^{hk}),\varepsilon(v^h)\right)_{\mathcal{K}}+\int_{\Gamma_C}p_\nu\left(\left[u_{\nu_n}^{hk}\right]_g\right)v_\nu^h\,da+\int_{\Gamma_C}\left(\lambda_{\nu_n}^{hk}-r\left(u_{\nu_n}^{hk}-g^h\right)\right)_-v_\nu^h\,da$$
$$-\int_{\Gamma_C}P_{B(p_\tau(u_\tau))}\left(\lambda_{\tau_n}^{hk}-ru_{\tau_n}^{hk}\right).v_\tau^h\,da=(f_n,v^h)_V-\left(\frac{k}{2}\,\mathcal{R}(t_n-t_{n-1})\varepsilon(u_{n-1}^{hk}),\varepsilon(v^h)\right)_{\mathcal{K}} \tag{20}$$
$$-\left(k\sum_{j=1}^{n-1}\mathcal{R}(t_n-t_j)\varepsilon\left(u_j^{hk}\right),\varepsilon(v^h)\right)_{\mathcal{K}}-\left(\frac{k}{2}\,\mathcal{R}(t_n-t_0)\varepsilon(u_0^{hk}),\varepsilon(v^h)\right)_{\mathcal{K}}\qquad\forall v^h\in V^h,$$

$$-\frac{1}{r}\int_{\Gamma_C}\left(\lambda_{\nu_n}^{hk}+\left(\lambda_{\nu_n}^{hk}-r\left(u_{\nu_n}^{hk}-g^h\right)\right)_-\right)\gamma_\nu^h\,da$$
$$-\frac{1}{r}\int_{\Gamma_C}\left(\lambda_{\tau_n}^{hk}-P_{B(p_\tau(u_\tau))}\left(\lambda_{\tau_n}^{hk}-ru_{\tau_n}^{hk}\right)\right).\gamma_\tau^h\,da=0\qquad\forall\gamma^h\in W^h, \tag{21}$$

where $[.]_g:\mathbb{R}\to\mathbb{R}$ is the function defined by $[s]_g=\begin{cases}s, & s\le g\\ 0, & s>g\end{cases}$ and $(.)_-$ is the negative part ($(s)_-=max\{0,-s\}$); also, $P_{B(p_\tau(u_\tau))}$ is the projection on the ball B of center 0 and radius $p_\tau(u_\tau)$ and r is positive penalty coefficient.

The numerical approximation of Problem P_V^{hk} leads to the solution of a system of nonlinear equations. Next, the three unknowns $(u,\lambda_\nu,\lambda_\tau)$ of this nonlinear system is computed by using a generalized Newton method which leads, at each iteration, to the solution of a linear system, see [2], [12] and [13] for details.

Finally, the discrete contact problems are solved employing the open-source finite element library GetFEM++ (see [20]).

4. NUMERICAL SIMULATIONS

The physical setting is presented in Figure 1. Where, $\Omega=(0,1)\times(0,1)$ and $\Gamma_D=(\{0\}\times[0,1])\cup(\{1\}\times[0,1])$, $\Gamma_N=[0,1]\times\{1\}$ and $\Gamma_C=[0,1]\times\{0\}$. On the part $\{0\}\times[0,1]$, the body is clamped and therefore, the displacement field vanishes there; the horizontal component of the displacement field vanishes on the part $\{1\}\times[0,1]$. Vertical tractions act on the part Γ_N of the boundary. The body is in frictional contact with a foundation on its lower boundary Γ_C.
We model the material's behavior with a viscoelastic linear constitutive law in which the elasticity tensor $\mathcal{A}$ and the relaxation tensor $\mathcal{R}$ satisfies

$$(\mathcal{A}\tau)_{ij}=\frac{E\kappa}{1-\kappa^2}(\tau_{11}+\tau_{22})\delta_{ij}+\frac{E}{1+\kappa}\tau_{ij},\quad 1\le i,j\le 2,\qquad \tau\in S^2,$$
$$(\mathcal{R}\tau)_{ij}=\alpha\tau_{ij},\quad 1\le i,j\le 2,\qquad \tau\in S^2,$$

Here E is the Young modulus, κ the Poisson ratio of the material and δ_{ij} denotes the Kronecker symbol.
For the computation below we used the following data:

$$E=10^2\,N/m^2,\ \kappa=0.3,\ \alpha=10^2\,N/m^2\,s,\ f_0=(0,-10)\ N/m^2,\ f_N=(0,-20)\ N/m,$$
$$P_\nu(s)=r\,max\{0,s\},\ r=10^2\,N/m^2,\ g=-0.05\,m,\ P_\tau=\mu P_\nu,\ \mu=0.2,\ T=1\,s.$$

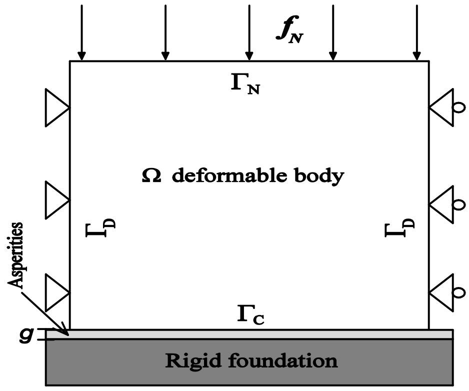

Figure 1. Physical setting

In Figure 2, we show the influence of the memory term on the deformation of the body. We observe that for $t = T = 1\ s$, the deformable body has recovered most of its original shape as a collateral effect of the memory. Then, in Figures. 3 and 4, we show the reactions and displacements of the nodes of the contact surface at final time $T = 1\ s$. The zone AB is a sliding zone formed by 13 nodes which are in a status of the normal compliance; there, the normal displacement is such that $0 < u_\nu < g$ and the tangential displacement does not vanishes, i.e., $u_\tau \neq 0$. In this zone, the friction follows the Coulomb law. The zone BC is a sliding zone formed of 19 nodes which are in a status of the unilateral condition; there, the normal displacement reaches the critical value of penetration g. In this zone, the friction follows the Tresca law with the friction bound F_b.

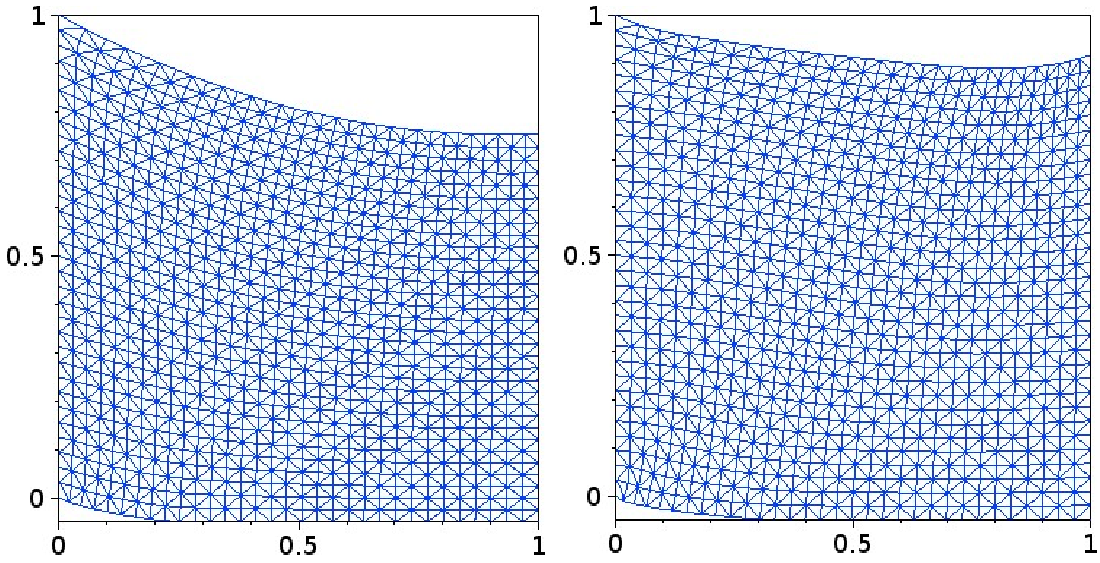

Figure 2. Deformed meshes for $t = 0.1$ s (left) and $t = T = 1$ s (right)

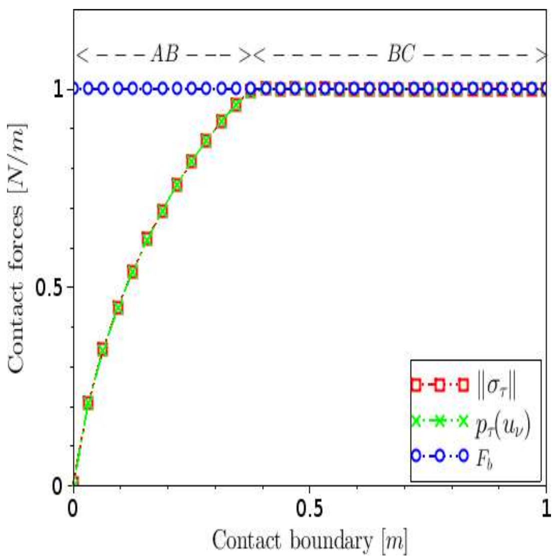

Figure 3. Frictional contact reactions on Γ_C

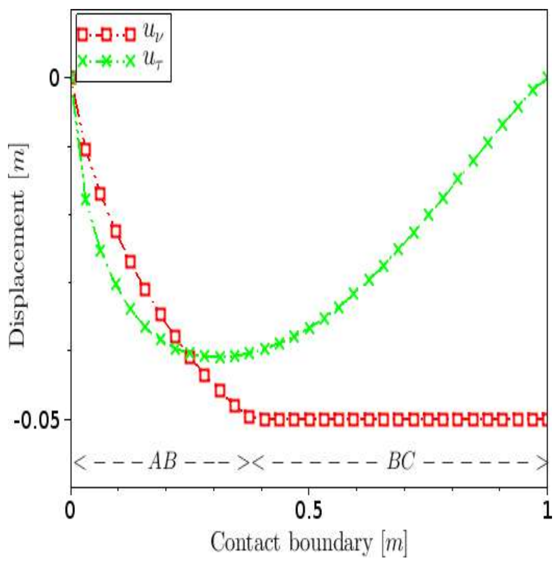

Figure 4. Displacement field on Γ_C

5. CONCLUSION

A new model of the contact process between a viscoelastic body and the foundation is numerically studied in this paper. The novelties arise in the fact that the material behavior is described by a viscoelastic constitutive law with long memory and the contact law with normal compliance and unilateral constraint is associated to a version of Coulomb's law of dry friction. A fully discrete scheme was used to approach the problem and a numerical algorithm which combine the penalty approach with the augmented Lagrangian method was implemented. Moreover, numerical simulations for a representative two-dimensional example were provided. We conclude that our simulations above underline the effects of the memory tem on the frictional contact process. Performing these simulations, we found that the numerical solution worked well and the convergence was rapid. This work opens the way to study further problems with other boundary contact conditions including dynamic effects

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