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Titchmarsh Theorems and K-Functionals for the Two-Sided Quaternion Fourier Transform

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ABSTRACT

The purpose of this paper is to study the Quaternion Fourier transforms of functions that satisfy Lipschitz conditions of certain orders. Thus we study the Quaternion Fourier transforms of Lipschitz function in the functions space $L^r(\mathbb{R}^2, \mathcal{H})$, where $\mathcal{H}$ a quaternion algebra which will be specified in due course. Our investigation into the problem was motivated by a theorem proved by Titchmarsh [[29], Theorem 85] for Lipschitz functions on the real line. we will give also some results on calculation of the K-functional which have number of applications of interpolation theory. In particular some recent problems in image processing and singular integral operators require the computation of suitable K-functionals. In this paper we will give some results concerning the equivalence of a K-functional and the modulus of smoothness constructed by the Steklov function.

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1. INTRODUCTION

The quaternionic Fourier transform (QFT) plays a vital role in the representation of signals. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency domain signal. The four QFT components separate four cases of symmetry in real signals instead of only two in the complex FT [8, 16]. In addition, understanding the QFT paves the way for understanding other integral transform, such as the Quaternion Fractional Fourier transform (QFRFT) [10, 23, 31, 17, 18], Quaternion linear canonical transform (QLCT) [24] and Quaternion Wigner-Ville distribution [6]. Due to the non-commutativity of multiplication of quaternions, there are different types of QFTs and we focus on the right-sided QFT (RQFT) and two-sided QFT (SQFT).

Recently it has become popular to generalize the Fourier transform (FT) from real and complex numbers [7] to quaternion algebra. In these constructions many FT properties still hold, others are modified. Therefore it is not a surprise that Titchmarsh's theorem also hold for the QFT. To the best of our knowledge, Titchmarsh's theorem, the equivalence of a K-functional and the modulus of smoothness for the QFT have not derived yet. In this Paper is extended the Titchmarsh's theorem in the frame of quaternion analysis and it is proved the equivalence of a K-functional and the modulus of smoothness using the Steklovs function. Recall that the relation between smoothness conditions imposed on functions $f(x)$ and the behavior of its Fourier transforms f near infinity is well known in the literature. In fact, a classical result of Titchmarsh [29] says that for $0 < \alpha \leq 1$, $1 < r \leq 2$ and

$$\left(\int_{\mathbb{R}} |f(x+h) - f(x)|^r dx\right)^{1/r} = O(h^\alpha)$$

where $h \longrightarrow 0$, then the Fourier transform $\widehat{f}$ belongs to $L^\beta(\mathbb{R})$, for

$$\frac{r}{r + \alpha r - 1} < \beta \leq \frac{r}{r-1}.$$

This theorem was extended to higher differences of functions in one and several variables in [33] and [34].
On the other hand, Younis in [32] studied the same phenomena for the wider Dini-Lipschitz class as well as for some other allied classes of functions. More precisely, he proved that if $f \in L^r(\mathbb{R})$, with $1 < r \leq 2$, such that

$$\left(\int_{\mathbb{R}} |f(x+h) - f(x)|^r dx\right)^{1/r} = O\left(\frac{h^\alpha}{log(\frac{1}{h})^\gamma}\right),$$

where $0 < \alpha \leq 1$, as $h \to 0$, then its Fourier transform $\widehat{f}$ belongs to $L^\beta(\mathbb{R})$, for

$$\frac{r}{r + \alpha r - 1} < \beta \leq r' = \frac{r}{r-1}$$

and

$$\frac{1}{\beta} < \gamma.$$

In recent years, these two results have been generalized in several different versions and for several different types of transforms (for example, see [3, 12, 13]).
In addition, the usual translation operator τ_h given by $\tau_h f(x) = f(x+h)$ plays a key role in the construction of modulus of continuity and smoothness which can be considered as a critical elements of direct and inverse theorems in approximation theory.
It is commonly known that studying the relation which exists between the smoothness properties of a function and the best approximations of this function in weight functional spaces is more convenient than usual with various generalized modulus of smoothness (see [26, 27]).
The K-functionals introduced by J. Peetre [25] take a part in a many problems of theory of approximation of functions. The study of the relation which exists between the modulus of smoothness and K-functionals is known as one of the major problems in the theory of approximation of functions . For many generalized modulus of smoothness these problems are studied, for example, in [4, 11, 14].

In order to describe our results, we first need to introduce some facts about harmonic analysis related to (two-sided) Quaternion Fourier transform (QFT). We cite here, as briefly as possible, some properties. For more details we refer to [1, 2, 21, 22, 15, 23, 30, 31, 5, 10, 20].

The quaternion algebra $\mathcal{H}$ was first invented by W. R. Hamilton in 1843 for extending complex numbers to a 4D algebra [28]. A quaternion $q \in \mathcal{H}$ can be written in this form

$$q = q_0 + \underline{q} = q_0 + iq_1 + jq_2 + kq_3$$

where i, j, k satisfy Hamilton's multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \;\; ij = -ji = k.$$

$$jk = -kj = i, \;\; ki = -ik = j.$$

Using Hamilton's multiplication rules, the multiplication of two quaternion $p = p_0 + \underline{p}$ and $q = q_0 + \underline{q}$ can be expressed as

$$pq = p_0 q_0 + p_0 \underline{q} + q_0 \underline{p} + \underline{pq}.$$

We define so, the conjugation of $q \in \mathcal{H}$ by $\overline{q} = q_0 - iq_1 - jq_2 - kq_3$. Clearly, $q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$. So the modulus of a quaternion q is defined by

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

In this paper, we study the quaternion-valued signal $f : \mathbb{R}^2 \to \mathcal{H}$ that can be expressed as

$$f = f_0 + if_1 + jf_2 + kf_3$$

where $x = x_1e_1 + x_2e_2 \in \mathbb{R}^2$ and $f_0,\ f_1,\ f_2$ and f_3 are real-valued functions. For $1 \leq r < \infty$, the quaternion modulus $L^r(\mathbb{R}^2, \mathcal{H})$ are defined as

$$L^r = L^r(\mathbb{R}^2, \mathcal{H}) = \{f/f : \mathbb{R}^2 \to \mathcal{H},\ \|f\|^r_{L^r(\mathbb{R}^2,\mathcal{H})} = \int_{\mathbb{R}^2} |f(x)|^r dx < \infty\ \}.$$

Let $f \in L^r(\mathbb{R}^2, \mathcal{H})$. The quaternion Fourier transform QFT of f is defined by

$$\mathcal{F}_Q(f)(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix_1\omega_1} f(x) e^{-jx_2\omega_2} dx.$$

The inner product of $f,\, g \in L^2(\mathbb{R}^2, \mathcal{H})$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{R}^2} f(x)\overline{g(x)} dx.$$

Clearly, $\|f\|_2^2 = \langle f, f\rangle$.

Now, we define $|.|_Q$ for $\mathcal{F}_Q(f)$ as

$$|\mathcal{F}_Q(f)(\omega)|_Q = \left(|\mathcal{F}_Q(f_0)(\omega)|^2 + |\mathcal{F}_Q(f_1)(\omega)|^2 + |\mathcal{F}_Q(f_2)(\omega)|^2 + |\mathcal{F}_Q(f_3)(\omega)|^2\right)^{1/2}.$$

For $f \in L^1(\mathbb{R}^2, \mathcal{H})$, we have

$$\|\mathcal{F}_Q(f)\|_{Q,\infty} \leq \|f\|_1. \tag{1}$$

(**Inversion formula**) The (two-sided) inverse QFT of $g \in L^1(\mathbb{R}^2, \mathcal{H})$

$$\mathcal{F}_Q^{-1}(g)(\omega) = \int_{\mathbb{R}^2} e^{jx_2\omega_2} g(x) e^{ix_1\omega_1} dx. \tag{2}$$

(**QFT Plancherel**) If $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^2(\mathbb{R}^2, \mathcal{H})$, then

$$\|\mathcal{F}_Q(f)\|_{Q,2} = \|f\|_2. \tag{3}$$

(**Hausdorff-Young inequality**) If $1 \leq r < 2$ and letting r' be such that $1/r + 1/r' = 1$ then for all $f \in L^r(\mathbb{R}^2, \mathcal{H})$ it holds that

$$\|\mathcal{F}_Q(f)\|_{Q,r'} \leq \|f\|_r. \tag{4}$$

Suppose that $\mathcal{F}_Q(f) \in L^1(\mathbb{R}^2, \mathcal{H})$ and $\dfrac{\partial^{n+m} f}{\partial x_1^n \partial x_2^m} \in L^1(\mathbb{R}^2, \mathcal{H})$. Then

$$\mathcal{F}_Q(\frac{\partial^{n+m} f}{\partial x_1^n \partial x_2^m})(w) = (iw_1)^n \mathcal{F}_Q(f)(w)(jw_2)^m,\ \ \forall n \in \mathbb{N}. \tag{5}$$

(**Shift property**) For a quaternion function $f \in L^1(\mathbb{R}^2, \mathcal{H})$, we denote by $\tau_k f(x)$ the shifted (translated) function defined by $\tau_k f(x) = f(x-k)$, where $k = k_1e_1 + k_1e_1 \in \mathbb{R}^2$. Then we obtain

$$\mathcal{F}_Q\{\tau_k f\}(\omega) = e^{ik_1\omega_1} \mathcal{F}_Q\{f\}(\omega_1, \omega_2) e^{jk_2\omega_2}. \tag{6}$$

For a function f on $L^1(\mathbb{R}^2, \mathcal{H})$ and for any $h_1, h_2 \in \mathbb{R}$, we define the operator Δ_{h_1,h_2} by

$$\Delta_{h_1,h_2} f(x) = f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) - f(x_1, x_2 + h_2) + f(x_1, x_2). \tag{7}$$

Definition 1 *Let $f(x) = f(x_l, x_2)$ belongs to $L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r < 2$. We say that f is in the Lipschitz space Lip(α_1, α_2, r) if*

$$\|\Delta_{h_1,h_2} f(x)\|_r = O(h_1^{\alpha_1} h_2^{\alpha_2}), \tag{8}$$

as h_1, h_2 tend to zero, $1 \leq r < \infty$, $0 < \alpha_1, \alpha_2 \leq 1$.

In $L^2(\mathbb{R}^2, \mathcal{H})$, consider the operator

$$\varphi_h f(x_1, x_2) = \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x_1 + \xi, x_2 + \eta) d\xi d\eta. \tag{9}$$

Let the function $f \in L^2(\mathbb{R}^2, \mathcal{H})$. The finite differences of the order m ($m \in 1, 2, 3, ...$) are defined as follows:

$$\Delta_h^m f(x_1, x_2) = (I - \varphi_h)^m f(x_1, x_2),$$

here I is the unit operator, and the mth-order generalized continuity modulus of the function f is defined by the formula

$$e_m(f, \delta)_2 = \sup_{0<h\leq\delta} \|\Delta_h^m f\|_2,$$

where $\delta > 0$.
Consider in $L^2(\mathbb{R}^2, \mathcal{H})$ the operator

$$Df(x) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) f(x), \tag{10}$$

and $D^0 f = f$, $D^r f = D(D^{r-1} f)$, $r = 1, 2, ...$
In view of formulas (5) and (10), we have

$$\mathcal{F}_Q\{Df\}(w) = -\left(w_1^2 + w_2^2\right) \mathcal{F}_Q\{f\}(w),$$

and hence

$$\mathcal{F}_Q\{D^r f\}(w) = (-1)^r \left(w_1^2 + w_2^2\right)^r \mathcal{F}_Q\{f\}(w). \tag{11}$$

This leads to the following definition of K-functionals: for $t > 0$

$$K_m(f, t)_2 = \inf \left\{ \|f - g\|_2 + t\|D^m g\|_2, \ \ g \in W_2^m \right\},$$

where W_2^m is the Sobolev space constructed by the operator D,

$$W_2^m = \left\{ f \in L^2(\mathbb{R}^2, \mathcal{H}), D^r f \in L^2(\mathbb{R}^2, \mathcal{H}), r = 1, ..., m \right\}.$$

2. ON THE TITCHMARSH THEOREM FOR THE QUATERNION FOURIER TRANSFORM

Lipschitz classes have been constantly employed in Fourier analysis, although they appear in the realm of trigonometric series , more than they occur in Fourier transforms. There are several new results in this section including theorems for, higher differences more precisely we will give some results associated with Dini-Lipschitz Functions in $L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r < 2$ for quaternion Fourier Transform. We here prove the following

Theorem 1 *Let f belongs to $L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r < 2$, and let f also belongs to Lip(α_1, α_2, r). Then $|\mathcal{F}_Q(f)(\omega)|_Q$ belongs to L^β, where*

$$\frac{r}{r + \alpha_s r - 1} < \beta \leq \frac{r}{r-1}, \ \ s = 1, 2.$$

Proof. By (7), we can show that the transform of $\Delta_{h_1,h_2} f(x)$ is given by

$$\mathcal{F}_Q(\Delta_{h_1,h_2} f(x)) = \left(e^{i\omega_1 h_1} - 1\right) \mathcal{F}_Q\{f\}(\omega) \left(e^{j\omega_2 h_2} - 1\right). \tag{12}$$

indeed

$$\begin{aligned}\mathcal{F}_Q(f(x_1+h_1,x_2+h_2)) &= e^{i\omega_1 h_1}\mathcal{F}_Q\{f\}(\omega)e^{j\omega_2 h_2}\\ \mathcal{F}_Q(f(x_1+h_1,x_2)) &= e^{i\omega_1 h_1}\mathcal{F}_Q\{f\}(\omega)\\ \mathcal{F}_Q(f(x_1,x_2+h_2)) &= \mathcal{F}_Q\{f\}(\omega)e^{j\omega_2 h_2}\end{aligned}$$

thus

$$\begin{aligned}\mathcal{F}_Q(\Delta_{h_1,h_2}f(x)) &= e^{i\omega_1 h_1}\mathcal{F}_Q\{f\}(\omega)e^{j\omega_2 h_2} - e^{i\omega_1 h_1}\mathcal{F}_Q\{f\}(\omega) - \mathcal{F}_Q\{f\}(\omega)e^{j\omega_2 h_2} + \mathcal{F}_Q\{f\}\\ &= e^{i\omega_1 h_1}\mathcal{F}_Q\{f\}(\omega)(e^{j\omega_2 h_2}-1) - \mathcal{F}_Q\{f\}(\omega)(e^{j\omega_2 h_2}-1)\end{aligned}$$

which gives the desired result. As well, we can easily obtain that

$$e^{i\omega_1 h_1}-1 = 2ie^{i\frac{\omega_1 h_1}{2}}\sin\frac{\omega_1 h_1}{2}$$

and

$$e^{j\omega_2 h_2}-1 = 2je^{j\frac{\omega_2 h_2}{2}}\sin\frac{\omega_2 h_2}{2}.$$

Hence

$$|\mathcal{F}_Q(\Delta_{h_1,h_2}f(x))|_Q = 4|\sin(\frac{\omega_1 h_1}{2})||\mathcal{F}_Q\{f\}(\omega)|_Q|\sin(\frac{\omega_2 h_2}{2})|.$$

by (4) and (8) we get that

$$\int_{\mathbb{R}^2}|\sin(\frac{\omega_1 h_1}{2})|^{r'}|\sin(\frac{\omega_2 h_2}{2})|^{r'}|\mathcal{F}_Q\{f\}(\omega)|_Q^{r'}d\omega_1 d\omega_2 = O(h_1^{\alpha_1 r'}h_2^{\alpha_2 r'}). \tag{13}$$

We obtain

$$\int_0^{\frac{2}{h_1}}\int_0^{\frac{2}{h_2}}|\frac{\omega_1 h_1}{2}|^{r'}|\frac{\omega_2 h_2}{2}|^{r'}|\mathcal{F}_Q\{f\}(\omega)|_Q^{r'}d\omega_1 d\omega_2 = O(h_1^{\alpha_1 r'}h_2^{\alpha_2 r'}),$$

it follows that

$$\int_0^{\frac{2}{h_1}}\int_0^{\frac{2}{h_2}}|\omega_1\omega_2|^{r'}|\mathcal{F}_Q\{f\}(\omega)|_Q^{r'}d\omega_1 d\omega_2 = O(h_1^{(\alpha_1-1)r'}h_2^{(\alpha_2-1)r'}).$$

Thus

$$\int_0^{X}\int_0^{Y}|\omega_1\omega_2|^{r'}|\mathcal{F}_Q\{f\}(\omega)|_Q^{r'}d\omega_1 d\omega_2 = O(X^{(1-\alpha_1)r'}Y^{(1-\alpha_2)r'}).$$

Now, we need to introduce the function ψ defined by

$$\psi(X,Y) = \int_1^{X}\int_1^{Y}|\omega_1\omega_2|^{\beta}|\mathcal{F}_Q\{f\}(\omega)|_Q^{\beta}d\omega_1 d\omega_2.$$

By the Hlder inequality, for $\beta \le r'$ we get that

$$\psi(X,Y) = O(X^{1-\alpha_1\beta+\frac{\beta}{r}}Y^{1-\alpha_2\beta+\frac{\beta}{r}}),$$

so that

$$\int_1^{X}\int_1^{Y}|\mathcal{F}_Q\{f\}(\omega)|_Q^{\beta}d\omega_1 d\omega_2 = O(X^{1-\beta-\alpha_1\beta+\frac{\beta}{r}}Y^{1-\beta-\alpha_2\beta+\frac{\beta}{r}}).$$

This quantity is bounded as $X,Y\to\infty$ if $1-\beta-\alpha_1\beta+\frac{\beta}{r}<0$ and $1-\beta-\alpha_2\beta+\frac{\beta}{r}<0$, i.e.

$$\frac{r}{r+\alpha_1 r-1}<\beta \text{ and } \frac{r}{r+\alpha_2 r-1}<\beta,$$

and the proof is complete. ■

Theorem 2 *Let f belongs to $L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r < 2$, and let*

$$\|\Delta_{h_1,h_2} f(x)\|_r = O\left(\frac{h_1^{\alpha_1}}{\log(\frac{1}{h_1})^{\gamma_1}} \cdot \frac{h_2^{\alpha_2}}{\log(\frac{1}{h_2})^{\gamma_1}}\right), \ \ 0 < \alpha_1, \alpha_2 < 1, \ \ as \ \ h_1, h_2 \to \infty. \tag{14}$$

Then $|\mathcal{F}_Q(f)(\omega)|_Q \in L^\beta$ for

$$\frac{r}{r + \alpha_s r - 1} < \beta \leq \frac{r}{r-1}, \ \ \beta > \frac{1}{\gamma_s}, \ \ s = 1, 2. \tag{15}$$

Proof. By analogy with the proof of the Theorem (1), we can establish the following result

$$\int_{\mathbb{R}^2} |\sin(\frac{\omega_1 h_1}{2})|^{r'} |\sin(\frac{\omega_2 h_2}{2})|^{r'} |\mathcal{F}_Q\{f\}(\omega)|_Q^{r'} d\omega_1 d\omega_2 = \left(\frac{h_1^{\alpha_1 r'}}{\log(\frac{1}{h_1})^{\gamma_1 r'}} \cdot \frac{h_2^{\alpha_2 r'}}{\log(\frac{1}{h_2})^{\gamma_1 r'}}\right). \tag{16}$$

If $0 < \omega_s < \frac{2}{h_s}$, $s = 1, 2$, then $|\frac{\omega_s h_s}{2}| < A|\sin(\frac{\omega_s h_s}{2})|$, A being constant and therefore

$$\int_0^{\frac{2}{h_1}} \int_0^{\frac{2}{h_2}} |\omega_1 \omega_2|^{r'} |\mathcal{F}_Q\{f\}(\omega)|_Q^{r'} d\omega_1 d\omega_2 = \left(\frac{h_1^{(\alpha_1 - 1) r'}}{\log(\frac{1}{h_1})^{\gamma_1 r'}} \cdot \frac{h_2^{(\alpha_2 - 1) r'}}{\log(\frac{1}{h_2})^{\gamma_1 r'}}\right).$$

Thus

$$\int_0^{X} \int_0^{Y} |\omega_1 \omega_2|^{r'} |\mathcal{F}_Q\{f\}(\omega)|_Q^{r'} d\omega_1 d\omega_2 = \left(\frac{X^{(1-\alpha_1) r'}}{\log(X)^{\gamma_1 r'}} \cdot \frac{Y^{(1-\alpha_2) r'}}{\log(Y)^{\gamma_1 r'}}\right).$$

Let

$$\phi(X, Y) = \int_1^{X} \int_1^{Y} |\omega_1 \omega_2|^{\beta} |\mathcal{F}_Q\{f\}(\omega)|_Q^{\beta} d\omega_1 d\omega_2.$$

For $\beta \leq r'$ and by the Hölder inequality, we obtain

$$\phi(X, Y) = O\left(\frac{X^{1 - \alpha_1 \beta + \frac{\beta}{r}}}{\log(X)^{\gamma_1 \beta}} \frac{Y^{1 - \alpha_2 \beta + \frac{\beta}{r}}}{\log(Y)^{\gamma_2 \beta}}\right),$$

it follows that

$$\int_1^{X} \int_1^{Y} |\mathcal{F}_Q\{f\}(\omega)|_Q^{\beta} d\omega_1 d\omega_2 = O\left(\frac{X^{1 - \beta - \alpha_1 \beta + \frac{\beta}{r}}}{\log(X)^{\gamma_1 \beta}} \frac{Y^{1 - \beta - \alpha_2 \beta + \frac{\beta}{r}}}{\log(Y)^{\gamma_2 \beta}}\right),$$

and for the right hand of this estimate to be bounded as $X, Y \to \infty$ one must have

$$1 - \beta - \alpha_1 \beta + \frac{\beta}{r} < 0, \ \ -\gamma_1 \beta < -1$$

and

$$1 - \beta - \alpha_2 \beta + \frac{\beta}{r} < 0, \ \ -\gamma_2 \beta < -1,$$

then

$$\frac{r}{r + \alpha_s r - 1} < \beta, \ \ \beta > \frac{1}{\gamma_s}, \ \ s = 1, 2.$$

This completes the proof. ∎

In the next we study if the proceeding theorems are still valid if we replace the first difference $\Delta_{h_1,h_2} f$ with a difference of higher order. We put

$$\Delta_{h_1,h_2}^{n,m} f(x) = \sum_{k_2=0}^{m} \sum_{k_1=0}^{n} (-1)^{n-k_1} (-1)^{m-k_2} \binom{n}{k_1} \binom{m}{k_2} f(x_1 + k_1 h_1, x_2 + k_2 h_2). \tag{17}$$

Observe that $\Delta_{h_1,h_2}^{1,1} f(x) = \Delta_{h_1,h_2} f(x)$. We now generalize Theorem 1 as follows.

Theorem 3 *If $f \in L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r < 2$, and if*

$$\|\Delta^{n,m}_{h_1,h_2} f(x)\|_r = O(h_1^{\alpha_1} h_2^{\alpha_2}), \;\; 0 < \alpha_1 < n, 0 < \alpha_2 < m \;\; as \;\; h_1, h_2 \to \infty, \tag{18}$$

then $|\mathcal{F}_Q(f)(\omega)|_Q \in L^\beta$, where

$$\frac{r}{r + \alpha_s r - 1} < \beta \leq \frac{r}{r-1}, \;\; s = 1, 2.$$

Proof. For two fixed h_1 and h_2, the transform of $\Delta^{n,m}_{h_1,h_2} f(x)$ is given

$$\left(\sum_{k_1=0}^{n} (-1)^{n-k_1} \binom{n}{k_1} e^{ik_1\omega_1 h_1}\right) \mathcal{F}_Q\{f\}(\omega) \left(\sum_{k_2=0}^{m} (-1)^{m-k_2} \binom{m}{k_2} e^{jk_2\omega_2 h_2}\right). \tag{19}$$

We can easily get that

$$\sum_{k_1=0}^{n} (-1)^{n-k_1} \binom{n}{k_1} e^{ik_1\omega_1 h_1} = \left(e^{i\omega_1 h_1} - 1\right)^n = (2i)^n e^{in\frac{\omega_1 h_1}{2}} \left(\sin \frac{\omega_1 h_1}{2}\right)^n$$

and

$$\sum_{k_2=0}^{m} (-1)^{m-k_2} \binom{m}{k_2} e^{jk_2\omega_2 h_2} = \left(e^{j\omega_2 h_2} - 1\right)^m = (2j)^m e^{jm\frac{\omega_2 h_2}{2}} \left(\sin \frac{\omega_2 h_2}{2}\right)^m.$$

which yields

$$|\mathcal{F}_Q(\Delta^{n,m}_{h_1,h_2} f(x))|_Q = 2^{n+m} |\sin(\frac{\omega_1 h_1}{2})|^n |\mathcal{F}_Q\{f\}(\omega)|_Q |\sin(\frac{\omega_2 h_2}{2})|^m.$$

By (4) we obtain

$$\int_{\mathbb{R}^2} |\sin(\frac{\omega_1 h_1}{2})|^{nr'} |\sin(\frac{\omega_2 h_2}{2})|^{mr'} |\mathcal{F}_Q\{f\}(\omega)|_Q^{r'} d\omega_1 d\omega_2 = O(h_1^{\alpha_1 r'} h_2^{\alpha_2 r'}). \tag{20}$$

So that

$$\int_0^{\frac{2}{h_1}} \int_0^{\frac{2}{h_2}} |\omega_1|^{nr'} \omega_2|^{mr'} |\mathcal{F}_Q\{f\}(\omega)|_Q^{r'} d\omega_1 d\omega_2 = O(h_1^{(\alpha_1 - n)r'} h_2^{(\alpha_2 - m)r'}),$$

which gives the desired result. ■
We shall also generalize Theorem 2 to the following theorem

Theorem 4 *If f belongs to $L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r < 2$, and if*

$$\|\Delta^{n,m}_{h_1,h_2} f(x)\|_p = O\left(\frac{h_1^{\alpha_1}}{\log(\frac{1}{h_1})^{\gamma_1}} . \frac{h_2^{\alpha_2}}{\log(\frac{1}{h_2})^{\gamma_1}}\right), \;\; 0 < \alpha_1 < n, 0 < \alpha_2 < m \;\; as \;\; h_1, h_2 \to \infty, \tag{21}$$

then $|\mathcal{F}_Q(f)(\omega)|_Q \in L^\beta$, where (15) holds.

Proof. The proof for this theorem is very similar to the proof of Theorem (3)and then

$$\int_0^{\frac{2}{h_1}} \int_0^{\frac{2}{h_2}} |\omega_1|^{mr'} |\omega_2|^{mr'} |\mathcal{F}_Q\{f\}(\omega)|_Q^{p'} d\omega_1 d\omega_2 = \left(\frac{h_1^{(\alpha_1 - n)r'}}{\log(\frac{1}{h_1})^{\gamma_1 r'}} . \frac{h_2^{(\alpha_2 - m)r'}}{\log(\frac{1}{h_2})^{\gamma_1 r'}}\right).$$

The rest of the proof is now analogous to the proof of Theorem 2 and (15) holds. ■

3. EQUIVALENCE OF A K-FUNCTIONAL AND THE MODULUS OF SMOOTHNESS FOR QUATERNION FOURIER TRANSFORM

Modulus of smoothness represent important tools in obtaining quantitative estimates of the error of approximation for positive processes. There are many such special functions associated with wide classes of function spaces. On another hand , in many problems of the theory of approximation of functions the K-functionals play an important role. The study of the connection between the modulus of smoothness and K-functionals is one of the main problems in the theory of approximation of functions. For various generalized modulus of smoothness these problems are studied, for example, in [9, 19].
In order to prove the main result, we shall need some preliminary results.

Lemma 1 *If f belongs to $L^2(\mathbb{R}^2, \mathcal{H})$, then*

$$\|\Delta_h^m f\|_2 \leq 2^m \|f\|_2. \tag{22}$$

Proof. Using the inequality (9), we have $\|\varphi_h f\|_2 \leq \|f\|_2$. Then $\|\Delta_h^1 f\|_2 \leq 2\|f\|_2$. Thus the result follows easily by using the recurrence for m. ∎

Lemma 2 *If quaternion function $f \in L^2(\mathbb{R}^2, \mathcal{H})$, then*

$$\mathcal{F}_Q\{\varphi_h f\}(w) = \frac{\sin(w_1 h)}{w_1 h}\frac{\sin(w_2 h)}{w_2 h}\mathcal{F}_Q\{f\}(w). \tag{23}$$

Proof. Let $f \in L^2(\mathbb{R}^2, \mathcal{H})$, we have

$$\begin{aligned}\mathcal{F}_Q\{\varphi_h f\}(w) &= \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} e^{i\omega_1\xi}\mathcal{F}_Q\{f\}(\omega)e^{j\omega_2\eta}d\xi d\eta\\ &= \left(\frac{1}{2h}\int_{-h}^{h} e^{i\omega_1\xi}d\xi\right)\mathcal{F}_Q\{f\}(\omega)\left(\frac{1}{2h}\int_{-h}^{h} e^{j\omega_2\eta}d\eta\right).\end{aligned}$$

Since

$$\frac{1}{2h}\int_{-h}^{h} e^{i\omega_1\xi}d\xi = \frac{\sin(w_1 h)}{w_1 h}$$

and

$$\frac{1}{2h}\int_{-h}^{h} e^{j\omega_2\eta}d\eta = \frac{\sin(w_2 h)}{w_2 h}.$$

So that the transform of $\varphi_h f(x)$ is given as

$$\frac{\sin(w_1 h)}{w_1 h}\frac{\sin(w_2 h)}{w_2 h}\mathcal{F}_Q\{f\}(w).$$

Which gives the desired result. ∎

corollaire 1 *For any function f belongs to $L^2(\mathbb{R}^2, \mathcal{H})$, we have*

$$\mathcal{F}_Q\{\Delta_h^m f\}(w) = \left(1 - \frac{\sin(w_1 h)}{w_1 h}\frac{\sin(w_2 h)}{w_2 h}\right)^m \mathcal{F}_Q\{f\}(w). \tag{24}$$

Lemma 3 *Let f belongs to $L^2(\mathbb{R}^2, \mathcal{H})$ and $t > 0$. Then*

$$e_m(f,t)_2 \leq \frac{1}{6^m}t^{2m} \parallel D^m f \parallel_2 . \tag{25}$$

Proof. Let $h \in (0, t]$. By (11), (24) and the Parseval equality, we get

$$\parallel \Delta_h^m f \parallel_2 = \parallel \left(1 - \frac{\sin(w_1 h)}{w_1 h}\frac{\sin(w_2 h)}{w_2 h}\right)^m \mathcal{F}_Q\{f\}(w) \parallel_2 \tag{26}$$

and

$$\| D^m f \|_2 = \| \left(w_1^2 + w_2^2\right)^m \mathcal{F}_Q\{f\}\|_2. \tag{27}$$

Hence, by (26) and (27) it follows that

$$\| \Delta_h^m f \|_2 = h^{2m} \| \left(\frac{1 - \frac{\sin(w_1 h)}{w_1 h}\frac{\sin(w_2 h)}{w_2 h}}{w_1^2 + w_2^2} \right)^m \left(w_1^2 + w_2^2\right)^m \mathcal{F}_Q\{f\}(w) \|_2 .$$

Since

$$0 \leq 1 - \frac{\sin(y)}{y} \leq \frac{y^2}{6}, \ \ | \sin(y) | \leq | y |, \ \ y \in \mathbb{R},$$

and

$$1 - \frac{\sin(w_1 h)}{w_1 h}\frac{\sin(w_2 h)}{w_2 h} = \left(1 - \frac{\sin(w_1 h)}{w_1 h}\right)\frac{\sin(w_2 h)}{w_2 h} + \left(1 - \frac{\sin(w_2 h)}{w_2 h}\right),$$

we obtain

$$0 \leq 1 - \frac{\sin(w_1 h)}{w_1 h}\frac{\sin(w_2 h)}{w_2 h} \leq \frac{1}{6}\left((w_1 h)^2 + (w_2 h)^2\right).$$

Thus

$$\| \Delta_h^m f \|_2 \leq \frac{1}{6^m} h^{2m} \| \left(w_1^2 + w_2^2\right)^m \mathcal{F}_Q\{f\}(w) \|_2 .$$

This combined with (27), we have

$$\| \Delta_h^m f \|_2 \leq \frac{1}{6^m} h^{2m} \| D^m f \|_2 \leq \frac{1}{6^m} t^{2m} \| D^m f \|_2,$$

which gives the desired result. ■

Now, given $\nu > 0$, We introduce the following operator for $f \in L^2(\mathbb{R}^2, \mathcal{H})$ by

$$Q_\nu(f)(x) = \mathcal{F}_Q^{-1}\{\mathcal{F}_Q(f)(\omega)\mathbb{I}_\nu(\omega)\}(x),$$

where $\mathbb{I}_\nu(\lambda)$ is the characteristic function of the segment $[-\nu, \nu]$.

It is easy to show that the function $Q_\nu(f)$ is infinitely differentiable and belongs to all classes W_2^m, $m \in \{1, 2, 3, ...\}$.

Lemma 4 *If f belongs to $L^2(\mathbb{R}^2, \mathcal{H})$ and $\nu > 0$. Then there exists a positive constant c_2 such that*

$$\|f - Q_\nu(f)\|_2 \leq c_2 \|\Delta_{1/\nu}^m f\|_2. \tag{28}$$

Furthermore,

$$\|f - Q_\nu(f)\|_2 \leq c_2 e_m(f, 1/\nu)_2. \tag{29}$$

Proof. By Parseval equality, we get

$$\begin{aligned} \|f - Q_\nu(f)\|_2 &= \|(1 - \mathbb{I}_\nu(\omega))\mathcal{F}_Q(f)(\omega))\|_2 \\ = \| \frac{1 - \mathbb{I}_\nu(\omega)}{\left(1 - \frac{\sin(w_1/\nu)}{w_1/\nu}\frac{\sin(w_2/\nu)}{w_2/\nu}\right)^m} & \left(1 - \frac{\sin(w_1/\nu)}{w_1/\nu}\frac{\sin(w_2/\nu)}{w_2/\nu}\right)^m \mathcal{F}_Q\{f\}(w)\|_2. \end{aligned}$$

Now remark that $|w| \geq \nu$ implies $|w_1| \geq \frac{\nu}{\sqrt{2}}$ or $|w_2| \geq \frac{\nu}{\sqrt{2}}$, and since $| \sin(y) | \leq | y |$, we obtain

$$\left| \frac{\sin(w_1/\nu)}{w_1/\nu}\frac{\sin(w_2/\nu)}{w_2/\nu} \right| \leq \sup\left\{ \sup_{|y| \geq 1\sqrt{2}} |\operatorname{sinc}(y)|, \sup_{|y| \geq 1\sqrt{2}} |\operatorname{sinc}(y)| \right\} = c_1 < 1.$$

Thus

$$\sup_{w\in\mathbb{R}^2}\left\{\frac{1-\mathbb{I}_\nu(\omega)}{\left(1-\dfrac{\sin(w_1/\nu)}{w_1/\nu}\dfrac{\sin(w_2/\nu)}{w_2/\nu}\right)^m}\right\}\leq\frac{1}{(1-c_1)^m}=c_2.$$

Hence

$$\|f-Q_\nu(f)\|_2\leq c_2\|\left(1-\frac{\sin(w_1/\nu)}{w_1/\nu}\frac{\sin(w_2/\nu)}{w_2/\nu}\right)^m\mathcal{F}_Q\{f\}(w)\|_2\leq c_2\|\Delta^m_{1/\nu}f\|_2.$$

This completes the proof. ■

Lemma 5 *Let $f\in L^2(\mathbb{R}^2,\mathcal{H})$. Then there exists a positive constant c_3 such that*

$$\|D^m(Q_\nu(f))\|\leq c_3\nu^{2m}\|\Delta^m_{1/\nu}f\|_2, \tag{30}$$

where $\nu>0$, $m\in\{1,2,...\}$. Furthermore,

$$\|D^m(Q_\nu(f))\|\leq c_3\nu^{2m}e_m(f,1/\nu)_2. \tag{31}$$

Proof. By the Parseval equality, we have

$$\begin{aligned}\|D^m(Q_\nu(f))\|_2 &= \|\mathcal{F}_Q\{D^m(Q_\nu(f))\}\|_2=\|\mathbb{I}_\nu(\omega)\mathcal{F}_Q\{f\}\|_2,\\ &= \|\left(w_1^2+w_2^2\right)^m\mathbb{I}_\nu(\omega)\mathcal{F}_Q\{f\}\|_2\end{aligned}$$

$$=\|\frac{\left(w_1^2+w_2^2\right)^m\mathbb{I}_\nu(\omega)}{\left(1-\dfrac{\sin(w_1/\nu)}{w_1/\nu}\dfrac{\sin(w_2/\nu)}{w_2/\nu}\right)^m}\left(1-\frac{\sin(w_1/\nu)}{w_1/\nu}\frac{\sin(w_2/\nu)}{w_2/\nu}\right)^m\mathcal{F}_Q\{f\}(w)\|_2.$$

Note that

$$\begin{aligned}\sup_{w\in\mathbb{R}^2}\left\{\frac{\left(w_1^2+w_2^2\right)^m\mathbb{I}_\nu(\omega)}{\left(1-\dfrac{\sin(w_1/\nu)}{w_1/\nu}\dfrac{\sin(w_2/\nu)}{w_2/\nu}\right)^m}\right\} &= \nu^{2m}\sup_{|w|\leq\nu}\left\{\frac{\left((\dfrac{w_1}{\nu})^2+(\dfrac{w_2}{\nu})^2\right)^m}{\left(1-\dfrac{\sin(w_1/\nu)}{w_1/\nu}\dfrac{\sin(w_2/\nu)}{w_2/\nu}\right)^m}\right\},\\ &= \nu^{2m}\sup_{|\xi|\leq1}\left\{\frac{\left(\xi_1^2+\xi_2^2\right)^m}{\left(1-\dfrac{\sin(\xi_1)}{\xi_1}\dfrac{\sin(\xi_2)}{\xi_2}\right)^m}\right\}.\end{aligned}$$

Let

$$c_3=\sup_{|\xi|\leq1}\left\{\frac{\left(\xi_1^2+\xi_2^2\right)^m}{\left(1-\dfrac{\sin(\xi_1)}{\xi_1}\dfrac{\sin(\xi_2)}{\xi_2}\right)^m}\right\},$$

which is exactly (30), and it is clear that the formula (30) yields the inequality (31). ■

Theorem 5 *Let f belongs to $L^2(\mathbb{R}^2,\mathcal{H})$ and $\delta>0$. One can find positive constant c_4 such that*

$$\frac{1}{2^m}e_m(f,\delta)_2\leq K_m(f,\delta^{2m})_2\leq c_4e_m(f,\delta)_2. \tag{32}$$

Proof. Let $h\in(0,\delta]$ and $g\in W_2^m$. Using Lemma 1 and 3, we deduce

$$\begin{aligned}\|\Delta^m_hf\|_2 &\leq \|\Delta^m_h(f-g)\|_2+\|\Delta^m_hg\|_2\\ &\leq 2^m\|f-g\|_2+\frac{1}{6^m}h^{2m}\parallel D^mg\parallel_2\\ &\leq 2^m(\|f-g\|_2+\delta^{2m}\parallel D^mg\parallel_2).\end{aligned}$$

Calculating the supremum with respect to $h \in (0, \delta]$ and the infimum with respect to all possible functions $g \in W_2^m$, we obtain

$$e_m(f, \delta)_2 \leq 2^m K_m(f, \delta^{2m})_2.$$

Since $Q_\nu(f) \in W_m^2$, by the definition of of a K-functional, it follows that

$$K_m(f, \delta^{2m})_2 \leq \|f - Q_\nu(f)\|_2 + \delta^{2m} \parallel D^m Q_\nu(f) \parallel_2 .$$

Therefore, using (29) and (31), we have

$$K_m(f, \delta^{2m})_2 \leq c_2 e_m(f, 1/\nu)_2 + c_3(\delta\nu)^{2m} e_m(f, 1/\nu)_2.$$

Putting $\nu = \dfrac{1}{\delta}$ in this inequality, we deduce

$$K_m(f, \delta^{2m})_2 \leq c_4 e_m(f, \delta)_2,$$

where $c_4 = c_2 + c_3$. This concludes the proof of Theorem 5. ■

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