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Harmonic analysis associated with the Weinstein type operator on $\mathbb{R}^d$

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ABSTRACT

We consider the Weinstein type operator $\Delta_{\alpha,d}$ on $\mathbb{R}^d$. We build transmutation operators $\mathcal{R}_\alpha$ which turn out to be transmutation operator between $\Delta_{\alpha,d}$ and the Laplacian Δ_d. Using this transmutation operators and its dual ${}^t\mathcal{R}_\alpha$, we develop a new commutative harmonic analysis on $\mathbb{R}^d$ corresponding to the operator $\Delta_{\alpha,d}$.

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1. INTRODUCTION

We consider the Weinstein operator on $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times [0, \infty[$ defined by

$$\Delta_\alpha = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}, \quad \alpha > -\frac{1}{2}. \tag{1}$$

The Weinstein operator has several applications in pure and applied mathematics especially in fluid mechanics (cf. [3]). The harmonic analysis associated with the Weinstein operator is studied by Ben Nahia and Ben Salem (cf. [1, 2]). In particular the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator. This transform is called the Weinstein transform. Very recently, many authors have investigated the behavior of the Weinstein transform with respect to several problems already studied for the Fourier transform; for instance, Paley-Wiener theorems [6], the Bockner-Hecke theorem [4], uncertainty principles associated with the Weinstein transform [5]. In this paper we introduce the Weinstein type operator on $\mathbb{R}^d$ defined by

$$\Delta_{\alpha,d} f(x) = \sum_{i=1}^{d} \frac{\partial^2 f(x', x_d)}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\left(\frac{\partial f(x', x_d)}{\partial x_d} - \frac{\partial f(x', 0)}{\partial x_d}\right)$$

where $\alpha > \frac{-1}{2}$, $x = (x', x_d) = (x_1, x_2, .., x_d) \in \mathbb{R}^d$. If $f(x', 0) = 0$ we regain the Weinstein operator defined by (1). Throughout this paper, we provide a new harmonic analysis on $\mathbb{R}^d$ corresponding to the Weinstein type operator $\Delta_{d,\alpha}$.

Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper, we denote by

- $a_\alpha = \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$, $\alpha > \frac{-1}{2}$. $d\mu_\alpha(x) = |x_d|^{2\alpha+1}dx_d dx' = |x_d|^{2\alpha+1}dx_1 dx_2 ... dx_d$.
- $E(\mathbb{R}^d)$ (resp. $D(\mathbb{R}^d)$) the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$, (resp. with compact support).
- $L^p_\alpha(\mathbb{R}^d)$ the class of measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}^d} |f(x',x_d)|^p |x_d|^{2\alpha+1} dx' dx_d\right)^{\frac{1}{p}} < +\infty, \text{ if } p < +\infty.$$
$$\|f\|_\infty = ess \sup_{x\in\mathbb{R}^d} |f(x)| < +\infty, \text{ if } p = +\infty.$$

- $f_e(x) = \dfrac{f(x',x_d) + f(x',-x_d)}{2}$ and $f_o(x) = \dfrac{f(x',x_d) - f(x',-x_d)}{2}$.

We recapitulate some facts about the eigenfunction $\Phi_{\lambda,\alpha}$ related to the Weinstein operator Δ_α. For more details we refer to [1].

For all $(x,\lambda) \in \mathbb{R}^d \times \mathbb{C}^d$ the eigenfunction $\Phi_{\lambda,\alpha}$ related to the Weinstein operator Δ_α as defined by

$$\Phi_{\lambda,\alpha}(x) = a_\alpha e^{-i\langle x',\lambda'\rangle} j_\alpha(x_d\lambda_d), \tag{2}$$

where $j_\alpha(z) = \Gamma(\alpha+1)\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n!\Gamma(n+\alpha+1)}$.

As $j_\alpha(x_d\lambda_d) = a_\alpha \displaystyle\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(x_d\lambda_d t)dt$,
it follows that

$$\Phi_{\lambda,\alpha}(x) = a_\alpha e^{-i\langle x',\lambda'\rangle} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(x_d\lambda_d t)dt.$$

We may now define the function $\Psi_{\lambda,\alpha}$ which is inspired by $\Phi_{\lambda,\alpha}$.
For all $(x,\lambda) \in \mathbb{R}^d \times \mathbb{C}^d$ we define the function $\Psi_{\lambda,\alpha}$ as follows

$$\Psi_{\lambda,\alpha}(x) = a_\alpha e^{-i\langle x',\lambda'\rangle} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-ix_d\lambda_d t}dt. \tag{3}$$

The function $\Psi_{\lambda,\alpha}$ has the following properties

i) $\Psi_{\lambda,\alpha}(x) = \Psi_{x,\alpha}(\lambda)$.

ii) for all $(x,\lambda) \in \mathbb{R}^d \times \mathbb{R}^d$ $|\Psi_{\lambda,\alpha}(x)| \le 1$.

iii) it follows from Riemann-Lebesgue lemma that $\lim_{\lambda_d\to\infty} \Psi_{\lambda,\alpha}(x) = 0$.

2. HARMONIC ANALYSIS ASSOCIATED WITH $\Delta_{D,\alpha}$

Proposition 1 $\Psi_{\lambda,\alpha}$ *satisfies the differential equation*

$$\Delta_{d,\alpha}\Psi_{\lambda,\alpha} = -\|\lambda\|^2 \Psi_{\lambda,\alpha}.$$

Proof. Notice that $\Psi_{\lambda,\alpha}(x) = e^{-i\langle x',\lambda'\rangle} \displaystyle\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-ix_d,\lambda_d t}dt$, it's clear that

$$\Delta_{d,\alpha}(e^{-i\langle x',\lambda'\rangle}) = -\sum_{i=1}^{d-1} \lambda_i^2.$$

On the other hand we have

$$\begin{aligned}\Delta_{d,\alpha}\left(\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-ix_d\lambda_d t}dt\right) &= -\lambda_d^2 \int_0^1 t^2(1-t^2)^{\alpha-\frac{1}{2}} e^{-ix_d\lambda_d t}dt \\ &- i\frac{2\alpha+1}{x_d}\lambda_d \int_0^1 t(1-t^2)^{\alpha-\frac{1}{2}}(e^{-ix_d\lambda_d t}-1)dt,\end{aligned}$$

but

$$\int_0^1 t(1-t^2)^{\alpha-\frac{1}{2}}(e^{-ix_d\lambda_d t}-1)dt = \frac{-i\lambda_d x_d}{2\alpha+1}\int_0^1 (1-t^2)^{\alpha+\frac{1}{2}} e^{-ix_d\lambda_d t}dt$$

it follows that

$$\begin{aligned}\Delta_{d,\alpha}\left(\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-ix_d\lambda_d t}dt\right) &= -\lambda_d^2 \int_0^1 \left(t^2(1-t^2)^{\alpha-\frac{1}{2}} + (1-t^2)^{\alpha+\frac{1}{2}}\right) \\ &\times\ e^{-ix_d\lambda_d t}dt \\ &= -\lambda_d^2 \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{-ix_d\lambda_d t}dt.\end{aligned}$$

Hence $\Delta_{d,\alpha}\Psi_{\lambda,\alpha} = -\|\lambda\|^2\Psi_{\lambda,\alpha}$. ■

Definition 1 *The Weinstein type intertwining operator is the operator $\mathcal{R}_\alpha$ defined on $E(\mathbb{R}^d)$ by*

$$\mathcal{R}_\alpha f(x) = a_\alpha \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} f(x', x_d t)dt. \tag{4}$$

Remark 1 *$\mathcal{R}_\alpha$ can be written in the following form*

$$\mathcal{R}_\alpha f(x) = a_\alpha sgn(x_d)|x_d|^{-2\alpha}\int_0^{x_d} (1-t^2)^{\alpha-\frac{1}{2}} f(x', t)dt,\ \ x_d \in \mathbb{R}^*.$$

Proposition 2 *For all $f \in E(\mathbb{R}^d)$ we have the following transmutation relation*

$$\Delta_{d,\alpha}(\mathcal{R}_\alpha f) = \mathcal{R}_\alpha(\Delta_d f),\ \textit{for all } f \in E(\mathbb{R}^d), \tag{5}$$

where Δ_d is the Laplacian on $\mathbb{R}^d$.

Proof. Notice that

$$\begin{aligned}\Delta_{d,\alpha}(\mathcal{R}_\alpha f)(x) &= \sum_{i=0}^{d-1}\frac{\partial^2\mathcal{R}_\alpha f(x',x_d)}{\partial x_d^2} + \frac{\partial^2\mathcal{R}_\alpha f(x',x_d)}{\partial x_d^2} \\ &+ \frac{2\alpha+1}{x_d}\left(\frac{\partial\mathcal{R}_\alpha f(x',x_d)}{\partial x_d} - \frac{\partial\mathcal{R}_\alpha f(x',0)}{\partial x_d}\right) \\ &= \sum_{i=0}^{d-1}\frac{\partial^2\mathcal{R}_\alpha f(x',x_d)}{\partial x_d^2} + \frac{\partial^2\mathcal{R}_\alpha f(x',x_d)}{\partial x_d^2} \\ &+ (2\alpha+1)\int_0^1 \frac{\partial^2\mathcal{R}_\alpha f(x',tx_d)}{\partial x_d^2}dt.\end{aligned}$$

Since

$$\frac{\partial^2\mathcal{R}_\alpha(f)(x)}{\partial x_d^2} = a_\alpha\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} t^2 \frac{\partial^2\mathcal{R}_\alpha f(x',tx_d)}{\partial x_d^2}dt$$

it follows that

$$\Delta_{d,\alpha}(\mathcal{R}_\alpha(f)(x)) = a_\alpha\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} t^2 \frac{\partial^2\mathcal{R}_\alpha f(x',tx_d)}{\partial x_d^2}dt + J(x)$$

where

$$J(x) = (2\alpha+1)a_\alpha \int_0^1 \int_0^1 (1-s^2)^{\alpha-\frac{1}{2}} s^2 \frac{\partial^2 \mathcal{R}_\alpha f(x', stx_d)}{\partial x_d^2} dt ds.$$

By Fubini's theorem

$$\begin{aligned} J(x) &= (2\alpha+1)a_\alpha \int_0^1 (1-s^2)^{\alpha-\frac{1}{2}} s \left(\int_0^s \frac{\partial^2 \mathcal{R}_\alpha f(x', tx_d)}{\partial x_d^2} dt \right) ds \\ &= (2\alpha+1)a_\alpha \int_0^1 \left(\int_t^1 (1-s^2)^{\alpha-\frac{1}{2}} s \right) ds \frac{\partial^2 \mathcal{R}_\alpha f(x', tx_d)}{\partial x_d^2} dt \\ &= a_\alpha \int_0^1 (1-t^2)^{\alpha+\frac{1}{2}} \frac{\partial^2 \mathcal{R}_\alpha f(x', tx_d)}{\partial x_d^2} dt \end{aligned}$$

hence

$$\begin{aligned} \Delta_{d,\alpha} \mathcal{R}_\alpha(f)(x) &= a_\alpha \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \sum_{i=0}^{d-1} \frac{\partial^2 \mathcal{R}_\alpha f(x', tx_d)}{\partial x_d^2} dt \\ &+ a_\alpha \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \frac{\partial^2 \mathcal{R}_\alpha f(x', tx_d)}{\partial x_d^2} dt \\ &= \mathcal{R}_\alpha(\Delta_d f)(x). \end{aligned}$$

■

Definition 2 *The dual of the Weinstein type intertwining operator $\mathcal{R}_\alpha$ is the operator ${}^t\mathcal{R}_\alpha$ defined on $D(\mathbb{R}^d)$ by*

$$ {}^t\mathcal{R}_\alpha(f)(y) = a_\alpha \int_{|y_d|}^{\infty} (s^2 - y_d^2)^{\alpha-\frac{1}{2}} f(y', sgn(y_d)s) s ds. \tag{6}$$

Proposition 3 *${}^t\mathcal{R}_\alpha$ satisfies for $f \in D(\mathbb{R}^d)$ and $g \in E(\mathbb{R}^d)$ the following relation*

$$\int_{\mathbb{R}^d} {}^t\mathcal{R}_\alpha(f)(y) g(y) dy = \int_{\mathbb{R}^d} f(y) \mathcal{R}_\alpha(g)(y) d\mu_\alpha(y). \tag{7}$$

Proof. Let $g \in E(\mathbb{R}^d)$ and $f \in D(\mathbb{R}^d)$, from Remarque 1 we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(y)\mathcal{R}_\alpha(g)(y) d\mu_\alpha(y) &= \int_{\mathbb{R}^{d-1}} \int_0^\infty a_\alpha x_d^{-2\alpha} \left(\int_0^{x_d} (x_d^2 - t^2)^{\alpha-\frac{1}{2}} g(x', t) dt \right) \\ &\times \ f(x)|x_d|^{2\alpha+1} dx_d dx' \\ &- \int_{\mathbb{R}^{d-1}} \int_0^\infty a_\alpha x_d^{-2\alpha} \left(\int_{x_d}^0 (x_d^2 - t^2)^{\alpha-\frac{1}{2}} g(x', t) dt \right) \\ &\times \ f(x)|x_d|^{2\alpha+1} dx_d dx' \end{aligned}$$

the result follows directly from Fubini's theorem, a change of variable and the Chasles Formula. ■

Definition 3 *The Weinstein type transform $\mathcal{F}_{\alpha,d}$ is defined on $L^1_\alpha(\mathbb{R}^d)$ by*

$$\mathcal{F}_{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}^d} f(x) \Psi_{\lambda,\alpha}(x) d\mu_\alpha(x), \text{ for all } \lambda \in \mathbb{R}^d. \tag{8}$$

Remark 2 *If f is an even function with the last variable then the Weinstein type transform $\mathcal{F}_{\alpha,d}$ coincides with the Weinstein transform $\mathcal{F}_\alpha$ defined by*

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}^d_+} f(x) \Phi_{\lambda,\alpha}(x) x_d^{2\alpha+1} dx_d dx', \text{ for all } \lambda \in \mathbb{R}^d.$$

Proposition 4 *Let f and g in $L^1_\alpha(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} \mathcal{F}_{\alpha,d}(f)(x)g(x)d\mu_\alpha(x) = \int_{\mathbb{R}^d} f(x)\mathcal{F}_{\alpha,d}(g)(x)d\mu_\alpha(x).$$

Proof. As f and g in $L^1_\alpha(\mathbb{R}^d)$, it follows from (ii) and Fubini's theorem that

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{F}_{\alpha,d}(f)(x)g(x)|d\mu_\alpha(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\lambda)g(x)\Psi_{\lambda,\alpha}(x)|d\mu_\alpha(\lambda)d\mu_\alpha(x) \\ &\leq \|f\|_{L^1_\alpha(\mathbb{R}^d)}\|g\|_{L^1_\alpha(\mathbb{R}^d)}.\end{aligned}$$

From (iii) and Fubini's theorem we deduce the desired result. ■

Theorem 1 *(Plancherel theorem) For all $f \in S(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 d\mu_\alpha(x) = C(\alpha)\left(\int_{\mathbb{R}^d} |\mathcal{F}_{\alpha,d}(f_e)(\lambda)|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}^d} |\mathcal{F}_{\alpha,d}(|f_o|)(\lambda)|^2 d\mu_\alpha(\lambda)\right),$$

where $S(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ and $C(\alpha) = \dfrac{1}{(2\pi)^{\frac{d}{2}}2^\alpha\Gamma(\alpha+1)}$.

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|^2 d\mu_\alpha(x) &= \int_{\mathbb{R}^d} |f_e(x) + f_o(x)|^2 d\mu_\alpha(x) \\ &= \int_{\mathbb{R}^d} |f_e(x)|^2 d\mu_\alpha(x) + 2\int_{\mathbb{R}^d} f_e(x)f_o(x)d\mu_\alpha(x) + \int_{\mathbb{R}^d} |f_o(x)|^2 d\mu_\alpha(x),\end{aligned}$$

as $f_e f_o$ is an odd function respect to the last variable, it follows by Fubini's theorem that

$$\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} f_e(x', x_d)f_o(x', x_d)|x_d|^{2\alpha+1}dx_d dx' = 0.$$

Since f_e and $|f_o|$ are even functions, then by Remarque 2 and Plancherel theorem for the Weinstein transform (see [5]) it follows that

$$\int_{\mathbb{R}^d} |f(x)|^2 d\mu_\alpha(x) = C(\alpha)\left(\int_{\mathbb{R}^d} |\mathcal{F}_{\alpha,d}(f_e)(\lambda)|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}^d} |\mathcal{F}_{\alpha,d}(|f_o|)(\lambda)|^2 d\mu_\alpha(\lambda)\right).$$

■

Proposition 5

- *For all $f \in L^1_\alpha(\mathbb{R}^d)$, the function $\mathcal{F}_\alpha(f)$ is continuous on $\mathbb{R}^d$ and we have*

$$\|\mathcal{F}_{\alpha,d}(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}.$$

- *For all $f \in S(\mathbb{R}^d)$ we have*

$$\mathcal{F}_\alpha(f)(y) = \mathcal{F}_0 \circ^t \mathcal{R}_\alpha(f)(y), \ \ \forall y \in \mathbb{R}^d$$

where $\mathcal{F}_0$ is the transformation defined by, for all $y \in \mathbb{R}^d$

$$\mathcal{F}_0(f)(y) = \int_{\mathbb{R}^d} f(x)e^{-i\langle y,x\rangle}dx, \ \ \forall f \in D(\mathbb{R}^d).$$

Proof.

- Since $|\Psi_{\lambda,\alpha}(x)| \leq 1$, it follows that

$$|f(x)\Psi_{\lambda,\alpha}(x)| \leq |f(x)|.$$

As $\lim_{\lambda_d\to\infty}\Psi_{\lambda,\alpha}(x) = 0$, it follows from the dominated convergence theorem that

$$\|\mathcal{F}_{W,\alpha}(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}.$$

•

$$\begin{aligned}\mathcal{F}_0 \circ^t \mathcal{R}_\alpha(f)(y) &= a_\alpha \int_{\mathbb{R}^{d-1}} e^{-i\langle x',y'\rangle} \int_0^\infty \int_{x_d}^\infty (s^2 - x_d^2)^{\alpha-\frac{1}{2}} x_d f(x',s) dx_d e^{-ix_d y_d} ds dx' \\ &+ a_\alpha \int_{\mathbb{R}^{d-1}} e^{-i\langle x',y'\rangle} \int_0^\infty \int_{x_d}^\infty (s^2 - x_d^2)^{\alpha-\frac{1}{2}} x_d f(x',-s) dx_d e^{ix_d y_d} ds dx'\end{aligned}$$

From Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{F}_0 \circ^t \mathcal{R}_\alpha(f)(y) &= a_\alpha \int_{\mathbb{R}^{d-1}} e^{-i\langle x',y'\rangle} \int_0^\infty \int_0^{x_d} (s^2 - x_d^2)^{\alpha-\frac{1}{2}} x_d f(x',s) dx_d e^{-ix_d y_d} ds dx' \\ &+ a_\alpha \int_{\mathbb{R}^{d-1}} e^{-i\langle x',y'\rangle} \int_0^\infty \int_0^{x_d} (s^2 - x_d^2)^{\alpha-\frac{1}{2}} x_d f(x',-s) dx_d e^{ix_d y_d} ds dx'.\end{aligned}$$

By a change of variable and using relation (3), we find that

$$\begin{aligned}\mathcal{F}_0 \circ^t \mathcal{R}_\alpha(f)(y) &= \int_{\mathbb{R}^{d-1}} \left(\int_0^\infty f(x)\Psi_{\lambda,\alpha}(x)|x_d|^{2\alpha+1} dx_d dx' \right) \\ &+ \int_{\mathbb{R}^{d-1}} \left(\int_0^\infty f(x)\Psi_{\lambda,\alpha}(x)|x_d|^{2\alpha+1} dx_d dx' \right)\end{aligned}$$

$$\mathcal{F}_0 \circ^t \mathcal{R}_\alpha(f)(y) = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^\infty f(x)\Psi_{\lambda,\alpha}(x)|x_d|^{2\alpha+1} dx_d dx' = \mathcal{F}_{\alpha,d}(f)(y).$$

■

3. APPLICATION

In this section we will give an application about Annihilating sets.

Definition 4 *Let $S,\ \Sigma$ be two measurable subsets of $\mathbb{R}^d$. Then (S,Σ) is called a weak annihilating pair for the Weinstein type transform if supp$f \subset S$ and supp$\mathcal{F}_\alpha(f) \subset \Sigma$ implies that $f = 0$, where supp $f = \{x : f(x) \neq 0\}$.*

Definition 5 *Let $S,\ \Sigma$ be two measurable subsets of $\mathbb{R}^d$. Then (S,Σ) is called a strong annihilating pair for the Weinstein type transform if there exists a constant $C(S,\Sigma)$ such that for all function $f \in L^2(\mathbb{R}^d,\mu_\alpha)$, with supp $\mathcal{F}_\alpha(f) \subset \Sigma$,*

$$\|f\|_{L^2(\mathbb{R}^d_+,\mu_\alpha)} \leq C(S,\Sigma)\|f\|_{L^2(S^c,\mu_\alpha)} \tag{9}$$

where $\mu_\alpha(x) = |x_d|^{2\alpha+1} dx' dx_\alpha$ and $S^c = \mathbb{R}^d_+ \backslash S$.

Theorem 2 *Let S and Σ be a pair of measurable subsets of $\mathbb{R}^d$ with $0 < \mu_\alpha(S)$, $\mu_\alpha(\Sigma) < \infty$, then the pair (S,Σ) is strong annihilating pair.*

Proof. The proof is similar to that of Theorem B in [7]. ■

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